Exercise-1

✍ **Marked questions are recommended for Revision.**
✍ **चिन्हित प्रश्न दोहराने योग्य प्रश्न है।**

PART - I : SUBJECTIVE QUESTIONS

भाग - I : विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

Section (A) : Integration using Standard Integral :
खण्ड (A) : मानक समाकल की सहायता से समाकलन :

A-1. Integrate with respect to x :
निम्नलिखित का x के सापेक्ष समाकलन कीजिए –

(i) $(2x + 3)^5$ (ii) $\sin 2x$ (iii) $\sec^2 (4x + 5)$

(iv) $\sec (3x + 2)$ (v) $\tan (2x + 1)$ (vi) 2^{3x+4}

(vii) $\frac{1}{2x+1}$ (viii) e^{4x+5}

Ans. (i) $\frac{(2x+3)^6}{12} + C$ (ii) $-\frac{\cos 2x}{2} + C$ (iii) $\frac{\tan(4x+5)}{4} + C$

(iv) $\frac{1}{3}\ \ell n\ |\sec (3x + 2) + \tan (3x + 2)\ | + C$ (v) $\frac{1}{2}\ \ell n\ |\sec(2x + 1)| + C$

(vi) $\frac{2^{3x+4}}{3\ \ell n\ 2} + C$ (vii) $\frac{1}{2}\ \ell n\ |2x + 1| + C$ (viii) $\frac{e^{4x+5}}{4} + C$

Sol. **(i)** $\int (2x+3)^5 dx = \frac{1}{2}\frac{(2x+3)^6}{6} + C = \frac{1}{12}(2x + 3)^6 + C$ (here put $(2x + 3) = t$ रखने पर)

(ii) $\int \sin 2x\ dx = -\frac{1}{2}\cos 2x + C$

(iii) $\int \sec^2(4x+5)\ dx = \frac{1}{4}\tan(4x + 5) + C$

(iv) $\int \sec(3x+2)\ dx = \frac{1}{3}\ \ell n\ [|\sec(3x+2)+\tan(3x+2)\ |] + C$

(v) $\int \tan(2x+1)\ dx = \frac{1}{2}\ell n\ |\sec(2x + 1)| + C$

(vi) $\int 2^{3x+4}\ dx = 16\int 2^{3x}\ dx = \frac{16}{3}\frac{2^{3x}}{\ell n 2} + C = \frac{2^{3x+4}}{3\ell n 2} + C$

(vii) $\int \frac{1}{2x+1}\ dx = \frac{1}{2}\ \ell n|(2x + 1)| + C$

(viii) $\int e^{4x+5}\ dx = \frac{e^{4x+5}}{4} + C$

A-2. Integrate with respect to x :
निम्नलिखित का x के सापेक्ष समाकलन कीजिए–

(i) $\sin^2 x$ (ii) $\cos^3 x$ (iii) $\sin 2x \cos 3x$

(iv) $4 \sin x \cos\frac{x}{2}\cos\frac{3x}{2}$ (v) $\frac{1}{\sqrt{x+3}-\sqrt{x+2}}$

Ans. (i) $\frac{x}{2} - \frac{1}{4}\sin 2x + C$ (ii) $\frac{\sin 3x}{12} + \frac{3}{4}\sin x + C$

(iii) $-\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$ (iv) $\cos x - \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x + C$

(v) $\frac{2}{3}\left((x+3)^{3/2} + (x+2)^{3/2}\right) + C$

Sol. (i) $\int \sin^2 x \; dx = \int \frac{1-\cos 2x}{2} \; dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

(ii) $\int \cos^3 x \; dx = \int \left(\frac{\cos 3x}{4} + \frac{3}{4}\cos x\right) dx = \frac{\sin 3x}{12} + \frac{3}{4}\sin x + C$

(iii) $\frac{1}{2}\int (\sin 5x - \sin x) \; dx = -\frac{\cos 5x}{10} + \frac{\cos x}{2} + C$

(iv) $\int 4 \sin x \frac{x}{2} \cos\cos\frac{3x}{2} \; dx = \int 2\sin x(\cos 2x + \cos x)dx$

$= \int 2\sin x \cos 2x \; dx + \int \sin 2x \; dx$

$= \int (\sin 3x - \sin x + \sin 2x) \; dx = \cos x - \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x + C$

(v) $I = \int \frac{1}{\sqrt{x+3} - \sqrt{x+2}} \; dx = \int \left(\frac{1}{\sqrt{x+3} - \sqrt{x+2}} \times \frac{\sqrt{x+3} + \sqrt{x+2}}{\sqrt{x+3} + \sqrt{x+2}}\right) dx$

$= \int \left[(x+3)^{1/2} + (x+2)^{1/2}\right] dx = \frac{2}{3}(x+3)^{3/2} + \frac{2}{3}(x+2)^{3/2} + C$

Section (B) : Integration using Substitution :

खण्ड (B) : प्रतिस्थापन की सहायता से समाकलन :

B-1. Integrate with respect to x :
निम्नलिखित का x के सापेक्ष समाकलन कीजिए–

(i) $x \sin x^2$ (ii) $\frac{x}{x^2+1}$ (iii) $\sec^2 x \tan x$ (iv) $\frac{e^x+1}{e^x+x}$

(v) $\frac{1-\sin x}{x+\cos x}$ (vi) $\frac{e^{2x}}{e^{2x}-2}$ (vii) $\frac{\cos 2x + x + 1}{x^2 + \sin 2x + 2x}$ (viii) $\frac{\sec x}{\ell n \; (\sec x + \tan x)}$

(ix) $\frac{x}{\sqrt{x+2}}$ (x) $\left(e^x + \frac{1}{e^x}\right)^2$ (xi) $(e^x + 1)^2 e^x$ (xii) $\frac{1}{x(x^5+1)}$

(xiii) $\frac{1}{x^5(1+x^5)^{\frac{1}{5}}}$ (xiv) $\frac{\sqrt{x^2-8}}{x^4}$

Ans. (i) $-\frac{1}{2}\cos x^2 + C$ (ii) $\frac{1}{2}\ell n |x^2 + 1| + C$

(iii) $\frac{1}{2}(\tan x)^2 + C$ or $\frac{\sec^2 x}{2} + C$ (iv) $\ell n |e^x + x| + C$

(v) $\ell n |x + \cos x| + C$ (vi) $\frac{1}{2}\ell n |e^{2x} - 2| + C$

(vii) $\frac{1}{2}\ell n |x^2 + \sin 2x + 2x| + C$ (viii) $\ell n | \ell n(\sec x + \tan x) | + C$

(ix) $\frac{2}{3}(x+2)^{3/2} - 4(x+2)^{1/2} + C$ (x) $\frac{1}{2}(e^{2x} - e^{-2x}) + 2x + C$

(xi) $\frac{1}{3}e^{3x} + e^{2x} + e^{x} + C$ (xii) $-\frac{1}{5}\ell n\left|1+\frac{1}{x^5}\right| + C$

(xiii) $-\frac{1}{4}\left(1+\frac{1}{x^5}\right)^{4/5} + C$ (xiv) $\frac{(x^2-8)^{3/2}}{24x^3} + C$

Sol. **(i)** $I = \int x\sin x^2 \, dx$ Put $x^2 = t$ रखने पर $\Rightarrow$ $x\,dx = \frac{dt}{2}$

$\Rightarrow$ $I = \frac{1}{2}\int \sin t \, dt = -\frac{\cos t}{2} + C = -\frac{\cos x^2}{2} + C$

(ii) $\int \frac{x}{x^2+1} dx$ Put $x^2 + 1 = t$ रखने पर $\Rightarrow$ $x\,dx = \frac{dt}{2}$

$\Rightarrow$ $I = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ell n(x^2+1) + C$

(iii) $I = \int \sec^2 x \tan x \, dx$ Put $\tan x = t$ रखने पर $\Rightarrow$ $\sec^2 x\,dx = dt$

$\Rightarrow$ $I = \int t \, dt = \frac{\tan^2 x}{2} + C$

(iv) $I = \int \frac{e^x+1}{e^x+x} dx$ Put $e^x + x = t$ रखने पर $\Rightarrow$ $(e^x+1)\,dx = dt$

$\Rightarrow$ $I = \ell n(e^x + x) + C$

(v) Let माना $x + \cos x = t \Rightarrow (1 - \sin x)\,dx = dt$

$I = \int \frac{dt}{t} = \ell n|t| + C = \ell n|x + \cos x| + C$

(vi) $I = \frac{1}{2}\int \frac{1}{t}dt$ where जहाँ $e^{2x} - 2 = t \Rightarrow 2e^{2x}\,dx = dt$

$= \frac{1}{2}\ell n|t| + C = \frac{1}{2}\ell n|e^{2x} - 2| + C$

(vii) Let माना $x^2 + \sin 2x + 2x = t$

$2(x + \cos 2x + 1)\,dx = dt$

$I = \int \frac{dt}{2t} = \frac{1}{2}\ell n|t| + C = \frac{1}{2}\ell n|x^2 + \sin 2x + 2x| + C$

(viii) Let माना $\ell n(\sec x + \tan x) = t \Rightarrow \sec x\,dx = dt$

$\therefore I = \int \frac{dt}{t} = \ell nt + C = \ell n|\ell n(\sec x + \tan x)| + C$

(ix) $\int \frac{x \, . \, dx}{\sqrt{x+2}}$

$= \int \sqrt{x+2}\, dx - \int \frac{2}{\sqrt{x+2}} dx$

$= \frac{2}{3}(x+2)^{3/2} - 2 \times 2\sqrt{x+2} + C$

$= \frac{2}{3}(x+2)^{3/2} - 4(x+2)^{1/2} + C$

(x) Let माना $e^x = t \Rightarrow e^x\,dx = dt \quad dx = \frac{1}{t}dt$

$$I = \int\left(t+\frac{1}{t}\right)^2\frac{1}{t}dt = \int\left(t+\frac{1}{t^3}+\frac{2}{t}\right)dt = \frac{t^2}{2}-\frac{1}{2t^2}+2\ell n\,|t|+C = \frac{1}{2}(e^{2x}-e^{-2x})+2x+C$$

(xi) Put $t = e^x$ रखने पर $\Rightarrow$ $dt = e^x\,dx$

$$= I=\int(t+1)^2\,dt = \int(t^2+2t+1)\,dt = \frac{t^3}{3}+t^2+t+C = \frac{e^{3x}}{3}+e^{2x}+e^x+C$$

(xii) $I = \int\frac{x^4}{x^5(x^5+1)}\,dx$ Let माना $x^5 = t \Rightarrow 5x^4\,dx = dt$

$$I = \frac{1}{5}\int\frac{dt}{t\ (t+1)} = \frac{1}{5}\int\left[\frac{1}{t}-\frac{1}{t+1}\right]dt$$

$$= \frac{1}{5}\ [\ell n|t| - \ell n\ |(t+1)|] + C$$

$$= \frac{1}{5}\ \ln\left(\frac{x^5}{1+x^5}\right)+C \quad = -\frac{1}{5}\ \ell n\left|1+\frac{1}{x^5}\right| + C$$

(xiii) $I = \int\frac{1}{x^5(1+x^5)^{1/5}}\,dx$

$$I = \int\frac{1}{x^6\left(\frac{1}{x^5}+1\right)^{1/5}}\,dx$$

Let $\left(\frac{1}{x^5}+1\right) = t$ रखने पर $\Rightarrow$ $\frac{-5}{x^6}dx = dt$

$$I = -\frac{1}{5}\int\frac{dt}{t^{1/5}}$$

$$= -\frac{1}{5}\,\frac{t^{-\frac{1}{5}+1}}{4/5} = -\frac{1}{4}\left(1+\frac{1}{x^5}\right)^{4/5}+C$$

(xiv) $I = \int\frac{\sqrt{x^2-8}}{x^4}\,dx$

$= \int\frac{1}{x^3}\sqrt{1-\frac{8}{x^2}}\,dx$ Put $1-\frac{8}{x^2} = t^2$ रखने पर $\Rightarrow$ $\frac{1}{x^3}dx = \frac{t}{8}\,dt$

$\Rightarrow$ $I = \int t.\frac{t}{8}dt$

$$= \frac{1}{8}\times\frac{t^3}{3} = \frac{1}{24}\left(1-\frac{8}{x^2}\right)^{3/2} = \frac{(x^2-8)^{3/2}}{24\ x^3}+C$$

B-2. Find the value of $\int\frac{d(x^2+1)}{\sqrt{(x^2+2)}}$. **[16JM120472]**

$\int \frac{d(x^2+1)}{\sqrt{(x^2+2)}}$ का मान होगा।

Ans. $2\sqrt{(x^2+2)}+C$

Sol. Let माना $I = \int \frac{d\ (x^2+1)}{\sqrt{x^2+2}} = \int \frac{2x}{\sqrt{x^2+2}}\ dx$

Put $x^2 + 2 = t$ रखने पर $\Rightarrow 2x\ dx = dt$

$\Rightarrow I = \int \frac{dt}{\sqrt{t}} = 2t^{1/2} + C = 2\ \sqrt{x^2+2} + C$

B-3. **Evaluate the following** :
निम्नलिखित को हल कीजिए –

(i) $\int \left(\frac{x\cos x - \sin x}{x \sin x}\right) dx$ **(ii)** $\int \left(\frac{\frac{x}{x+1} - \ln(x+1)}{x(\ln(x+1))}\right) dx$

Ans. (i) $\ln\left(\frac{\sin x}{x}\right) + C$ (ii) $\ln\left(\frac{\ln(x+1)}{x}\right) + C$

Sol. **(i)** $I = \int \left(\frac{x\cos x + \sin x - 2\sin x}{x\sin x}\right) dx = \int \frac{x\cos x + \sin x}{x \sin x} d - \int \frac{2}{x} dx = \ln(x \sin x) - 2 \ln x + C = \ln\left(\frac{\sin x}{x}\right) + C$

(ii) $I = \int \left(\frac{\frac{x}{x+1} + \ln(x+1) - 2\ln(x+1)}{x(\ln(x+1))}\right) dx = \int \left(\frac{\frac{x}{x+1} + \ln(x+1)}{x(\ln(x+1))}\right) dx - \int \frac{2}{x}\ dx = \ln\ (x\ \ln(x+1)) - \ln x^2 + C$

$= \ln\left(\frac{\ln(x+1)}{x}\right) + C$

Section (C) : Integration by parts :
खण्ड (C) : खण्डशः समाकलन :

C-1. Integrate with respect to x :
निम्नलिखित का x के सापेक्ष समाकलन कीजिए–

(i) $x\ \ell n\ x$ (ii) $x \sin^2 x$ (iii) $x \tan^{-1} x$ (iv) $\ell n\ x$

(v) $\sec^3 x$ (vi) $2x^3\ e^{x^2}$ (vii) $\sin^{-1}\sqrt{x}$ (viii) $\frac{x^2 \tan^{-1} x}{1+x^2}$

(ix) $e^x \sin x$ (x) $e^x (\sec^2 x + \tan x)$

Ans. (i) $\frac{x^2}{2}\ \ell n\ x - \frac{x^2}{4} + C$ (ii) $\frac{x^2}{4} - \frac{x}{4}\ \sin 2x - \frac{1}{8} \cos 2x + C$

(iii) $\frac{x^2}{2}\ \tan^{-1}x - \frac{x}{2} + \frac{1}{2}\ \tan^{-1}x + C$

(iv) $x\ (\ell nx - 1) + C$ (v) $\frac{\sec x \tan x}{2} + \frac{1}{2}\ \ell n\ |\sec x + \tan x| + C$

(vi) $(x^2 - 1)\ e^{x^2} + C$ (vii) $x \sin^{-1} \sqrt{x} + \dfrac{\sqrt{x}\sqrt{1-x}}{2} - \dfrac{1}{2} \sin^{-1} \sqrt{x} + C$

(viii) $x \tan^{-1}x - \dfrac{1}{2} \ell n\,(1 + x^2) - \dfrac{(\tan^{-1} x)^2}{2} + C$

(ix) $\dfrac{e^x}{2}$ (sinx – cosx) + C (x) $e^x \tan x + C$

Sol. (i) $\int x\ \ell n\ x\ dx = \ell nx.\dfrac{x^2}{2} - \int \dfrac{1}{x}.\dfrac{x^2}{2}dx = \ell nx.\dfrac{x^2}{2} - \dfrac{x^2}{4} + C$

(ii) $\int x\sin^2 x\ dx = \int x\left(\dfrac{1-\cos 2x}{2}\right) dx = \dfrac{x^2}{4} - \dfrac{1}{2}\int x\cos 2x\ dx$

$= \dfrac{x^2}{4} - \dfrac{x}{2} . \dfrac{\sin 2x}{2} + \dfrac{1}{2}\int \dfrac{\sin 2x}{2}\ dx + C = \dfrac{x^2}{4} - \dfrac{x}{4}\sin 2x - \dfrac{1}{8}\cos 2x + C$

(iii) $\int x\tan^{-1} x\ dx$

$= \tan^{-1} x.\dfrac{x^2}{2} - \int \dfrac{1}{1+x^2}.\dfrac{x^2+1-1}{2}dx = \dfrac{x^2}{2}\tan^{-1} x - \dfrac{1}{2}\int 1.dx + \dfrac{1}{2}\int \dfrac{1}{1+x^2}\ dx$

$= \dfrac{x^2}{2}\tan^{-1} x - \dfrac{x}{2} + \dfrac{1}{2}\tan^{-1} x + C$

(iv) $\int \ell nx\ dx = \int 1\ .\ \ell nx\ dx = x\ \ln x - \int \dfrac{1}{x}.x\ dx = x\ \ell nx - x + C$

(v) Let माना t = tanx

dt = sec²x dx

$I = \int \sqrt{1+\tan^2 x}\ \sec^2 x\, dx = \int \sqrt{1+t^2}\ dt = \dfrac{t}{2}\sqrt{1+t^2} + \dfrac{1}{2}\ \ell n\,|t + \sqrt{1+t^2}\,| + C$

$= \dfrac{\sec x \tan x}{2} + \ln|\sec x + \tan x| + C$

(vi) $I = \int 2x^3 e^{x^2} dx$ Put $x^2 = t$ रखने पर $\Rightarrow$ 2x dx = dt

$\Rightarrow$ $I = \int t.e^t\ dt = t.e^t - e^t + C = x^2.e^{x^2} - e^{x^2} + C$

(vii) $I = \int \sin^{-1}\sqrt{x}\ dx$ put $\sqrt{x} = t$ रखने पर $\Rightarrow \dfrac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow$ $I = 2\int \sin^{-1} t\ .\ t\ dt = 2\sin^{-1} t.\dfrac{t^2}{2} - 2\int \dfrac{1}{\sqrt{1-t^2}} \times \dfrac{t^2}{2}dt = t^2 \sin^{-1} t + \int \sqrt{1-t^2}\ dt - \int \dfrac{dt}{\sqrt{1-t^2}}$

$= t^2 \sin^{-1} t + \dfrac{t}{2}\sqrt{1-t^2} + \dfrac{1}{2}\sin^{-1} t - \sin^{-1} t + C = x \sin^{-1}\sqrt{x} + \dfrac{\sqrt{x}\ \sqrt{1-x}}{2} - \dfrac{1}{2}\sin^{-1}\sqrt{x} + C$

(viii) Let माना $\tan^{-1}x = t \Rightarrow \dfrac{1}{1+x^2}\ dx = dt$

$\Rightarrow$ $I = \int t.\tan^2 t\ dt = \int t(\sec^2 t - 1)\ dt = \int t.\ \sec^2 t\ dt - \int t\ dt$

$= t\tan t - \ell n\,|\sec t| - \dfrac{t^2}{2} + C$

$= x\tan^{-1}x - \dfrac{1}{2}\ \ell n(1 + x^2) - \dfrac{(\tan^{-1} x)^2}{2} + C$

(ix) $I = \int e^x \sin x \ dx$

$= -e^x \cos x + \int e^x \cos x \ dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \ dx$

$I = \frac{1}{2} e^x (\sin x - \cos x) + C$

(x) $\int e^x(\sec^2 x + \tan x) \ dx = e^x \tan x + C$

C-2. Find the antiderivative of $f(x) = \ln(\ln x) + (\ln x)^{-2}$ whose graph passes through (e, e). **[16JM120473]**
$f(x) = \ln(\ln x) + (\ln x)^{-2}$ का प्रतिअवकलज ज्ञात कीजिए जिसका वक्र (e, e) से गुजरता है।

Ans. $y = x\left[\ell n(\ell nx) - \frac{1}{\ell nx}\right] + 2e$

Sol. $\int f(x) \ dx = \int\left(\ell n \ (\ell nx) + \frac{1}{(\ell nx)^2}\right) dx$

Put $\ell nx = t$ रखने पर $\Rightarrow x = e^t \Rightarrow dx = e^t dt$

$= \int e^t\left(\ell nt + \frac{1}{t^2}\right) dt = \int e^t\left(\ell nt - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right) dt = e^t \left(\ell nt - \frac{1}{t}\right) + C \Rightarrow y = x\left(\ell n \ (\ell nx) - \frac{1}{\ell nx}\right) + C$

$\because$ it passes through (e, e) यह (e, e) से गुजरता है $\Rightarrow$ $C = 2e$

Hence, इस प्रकार $y = x\left[\ell n(\ell nx) - \frac{1}{\ell nx}\right] + 2e$

Section (D) : Algebraic integral :
खण्ड (D) : बीजगणितीय समाकलन :

D-1. Integrate with respect to x :
निम्नलिखित का x के सापेक्ष समाकलन कीजिए–

(i) $\frac{1}{x^2+4}$ (ii) $\frac{1}{x^2+5}$ (iii) $\frac{1}{x^2+2x+5}$

(iv) $\frac{2x+1}{x^2+3x+4}$ (v) $\frac{x^3-1}{x^3+x}$ (vi) $\frac{1}{\sqrt{x^2-4}}$

(vii) $\sqrt{x^2+4}$ (viii) $\sqrt{x^2+2x+5}$ (ix) $(x-1)\sqrt{1-x-x^2}$

(x) $x^5\sqrt{a^3+x^3}$

Ans. (i) $\frac{1}{2}\tan^{-1}\frac{x}{2} + C$ (ii) $\frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}} + C$ (iii) $\frac{1}{2}\tan^{-1}\left(\frac{(x+1)}{2}\right) + C$

(iv) $\ell n|x^2+3x+4| - \frac{4}{\sqrt{7}}\tan^{-1}\frac{2x+3}{\sqrt{7}} + C$ (v) $x - \arctan x + \ell n\frac{\sqrt{1+x^2}}{x} + C$

(vi) $\ell n|x + \sqrt{x^2-4}| + C$ (vii) $\frac{x}{2}\sqrt{x^2+4} + 2\,\ell n|x + \sqrt{x^2+4}| + C$

(viii) $\frac{x+1}{2}\sqrt{x^2+2x+5} + 2\,\ell n\left|x+1 + \sqrt{x^2+2x+5}\right| + C$

(ix) $-\frac{(1-x-x^2)^{3/2}}{3} - \frac{3}{8}(2x+1)\sqrt{1-x-x^2} - \frac{15}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + C$

(x) $\frac{2}{15}(a^3+x^3)^{5/2} - \frac{2a^3}{9}(a^3+x^3)^{3/2} + C$

Sol. **(i)** $\int \frac{1}{x^2+4}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$ **(ii)** $\int \frac{1}{x^2+5}dx = \frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}} + C$

(iii) $\int \frac{1}{(x+1)^2+4}dx = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$

(iv) $I = \int \frac{2x+1}{x^2+3x+4}\,dx$

Let माना $2x + 1 = A\left(\frac{d}{dx}(x^2+3x+4)\right) + B \Rightarrow 2x + 1 = A(2x + 3) + B \Rightarrow A = 1 \;\;\&\;\; B = -2$

$\Rightarrow \quad I = \int \frac{2x+3}{x^2+3x+4}\,dx - \int \frac{2}{x^2+3x+4}\,dx$

$= \ell n\left(x^2+3x+4\right) - \int \frac{2}{\left(x+\frac{3}{2}\right)^2+4-\frac{9}{4}}\,dx = \ell n(x^2 + 3x + 4) - \int \frac{2}{\left(x+\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\,dx$

$= \ell n(x^2 + 3x + 4) - \frac{4}{\sqrt{7}}\tan^{-1}\left(\frac{2x+3}{\sqrt{7}}\right) + C$

(v) $I = \int \frac{x^3-1}{x^3+x}\,dx = \int dx - \int \frac{dx}{1+x^2} - \int \frac{1}{x(x^2+1)}\,dx = x - \tan^{-1}x - \left[\int \frac{1}{x}dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx\right]$

$= x - \tan^{-1}x - \ell n|x| + \frac{1}{2}\ell n\,(x^2 + 1) + C = x - \tan^{-1}x + \ell n\left|\frac{\sqrt{1+x^2}}{x}\right| + C$

(vi) $\int \frac{1}{\sqrt{x^2-4}}\,dx = \ell n\left|\left(x+\sqrt{x^2-4}\right)\right| + C$

(vii) $\int \sqrt{x^2+4}\,dx$

$= \frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\ell n\left(\frac{x+\sqrt{x^2+4}}{2}\right) + C$

$= \frac{x}{2}\sqrt{x^2+4} + 2\ell n\left(x+\sqrt{x^2+4}\right) + C$

(viii) $\int \sqrt{x^2+2x+5}\,dx = \int \sqrt{(x+1)^2+2^2}\,dx$

$= \frac{(x+1)}{2}\sqrt{x^2+2x+5} + 2\ell n\left|x+1+\sqrt{x^2+2x+5}\right| + C$

(ix) $\int (x-1)\sqrt{1-x-x^2}\,dx$

$= \frac{1}{2}\int (2x+1-3)\sqrt{1-x-x^2}\,dx$

$= \frac{1}{2}\int (2x+1)\sqrt{1-x-x^2}\,dx - \frac{3}{2}\int \sqrt{1-x-x^2}\,dx$

$= \frac{1}{2}\int (2x+1)\sqrt{1-x-x^2}\,dx - \frac{3}{4}\int \sqrt{5-(2x+1)^2}\,dx$

$= -\frac{\left(1-x-x^2\right)^{3/2}}{3} - \frac{3}{8}(2x+1)\sqrt{1-x-x^2} - \frac{15}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + C$

(x) $I = \int x^5\sqrt{a^3+x^3}\,dx$ Put $a^3 + x^3 = t^2$ रखने पर $\Rightarrow$ $3x^2\,dx = 2t.dt$

$\Rightarrow \quad I = \int (t^2 - a^3)\ (t)\ \dfrac{2t\ dt}{3} = \dfrac{2}{3} \int (t^4 - a^3t^2)\ dt = \dfrac{2}{3}\left[\dfrac{t^5}{5} - \dfrac{a^3 \times t^3}{3}\right] + C$

$= \dfrac{2}{15}(a^3 + x^3)^{5/2} - \dfrac{2a^3}{9}(a^3 + x^3)^{3/2} + C$

D-2. Integrate with respect to x :
निम्नलिखित का x के सापेक्ष समाकलन कीजिए–

(i) $\dfrac{1}{(x+1)(x+2)}$ (ii)✍ $\dfrac{1}{(x^2+1)(x+3)}$

(iii) $\dfrac{3x+2}{(x+1)^2(x+2)}$ (iv) $\dfrac{1}{(x+1)(x+2)(x+3)}$

Ans. (i) $\ell n \left|\dfrac{x+1}{x+2}\right| + C$

(ii) $\dfrac{1}{10}\ \ell n\ |x + 3| - \dfrac{1}{20}\ \ell n\ |x^2 + 1| + \dfrac{3}{10}\ \tan^{-1} x + C$

(iii) $4\ell n|x + 1| + \dfrac{1}{(x+1)} - 4\ell n|x + 2| + C$

(iv) $\dfrac{1}{2}\ \ell n\ |x + 1| - \ell n\ |x + 2| + \dfrac{1}{2}\ \ell n\ |x + 3| + C$

Sol. **(i)** $\int \dfrac{1}{(x+1)(x+2)} = \int\left(\dfrac{1}{x+1} - \dfrac{1}{x+2}\right) dx = \ell n \left|\dfrac{x+1}{x+2}\right| + C$

(ii) $I = \int \dfrac{1}{(x^2+1)(x+3)} dx$

Let माना $\dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+1} = \dfrac{1}{(x^2+1)(x+3)} \quad \Rightarrow \quad A\ (x^2 + 1) + (Bx + C)(x + 3) = 1$

Put $x = -3$ रखने पर $\Rightarrow$ $A = 1/10$

and तथा $A + 3C = 1$ $A + B = 0$

$B = -1/10$
$C = 3/10$

$\Rightarrow I = \dfrac{1}{10}\int \dfrac{1}{x+3} dx - \dfrac{1}{20}\int \dfrac{2x}{x^2+1} dx + \dfrac{3}{10}\int \dfrac{1}{x^2+1} dx = \dfrac{1}{10}\ \ell n\ |x + 3| - \dfrac{1}{20}\ \ell n\ |x^2 + 1| + \dfrac{3}{10} \tan^{-1} x + K$

(iii) $I = \int \dfrac{3x+2}{(x+1)^2(x+2)} dx = \int \dfrac{A}{x+1} dx + \int \dfrac{B}{(x+1)^2} dx + \int \dfrac{C}{x+2} dx$

$= 4\int \dfrac{dx}{x+1} - \int \dfrac{dx}{(x+1)^2} - 4\int \dfrac{dx}{x+2} = 4\ell n|x + 1| + \dfrac{1}{(x+1)} - 4\ell n|x + 2| + C$

(iv) $I = \int \dfrac{1}{(x+1)(x+2)(x+3)}\ dx$ put $x + 1 = t$ रखने पर $\Rightarrow$ $dx = dt$

$\Rightarrow \quad I = \int \dfrac{1}{t\ (t+1)(t+2)}\ dt = \int \dfrac{1}{t}\left[\dfrac{1}{t+1} - \dfrac{1}{t+2}\right] dt = \int\left[\dfrac{1}{t(t+1)} - \dfrac{1}{t(t+2)}\right] dt$

$= \int\left[\left(\dfrac{1}{t} - \dfrac{1}{t+1}\right) - \dfrac{1}{2}\left(\dfrac{1}{t} - \dfrac{1}{t+2}\right)\right] dt = \dfrac{1}{2}\ \ell n|t| - \ell n|t + 1| + \dfrac{1}{2}\ \ell n\ |t + 2| + C$

$= \dfrac{1}{2}\ \ell n\ |x + 1| - \ell n\ |x + 2| + \dfrac{1}{2}\ \ell n\ |x + 3| + C$

D-3. Integrate with respect to x :

निम्नलिखित का x के सापेक्ष समाकलन कीजिए–

(i) $\dfrac{1}{x^4+x^2+1}$ (ii) $\dfrac{1+x^2}{1+x^4}$ (iii) $\dfrac{1-x^2}{1-x^2+x^4}$

Ans. (i) $\dfrac{1}{2\sqrt{3}}\tan^{-1}\left(\dfrac{x^2-1}{\sqrt{3}x}\right)-\dfrac{1}{4}\ell n\left|\dfrac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right|+C$

(ii) $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{x^2-1}{\sqrt{2}\ x}\right)+C$

(iii) $-\dfrac{1}{2\sqrt{3}}\ell n\left|\dfrac{x+\frac{1}{x}-\sqrt{3}}{x+\frac{1}{x}+\sqrt{3}}\right|+C$

Sol. **(i)** $I=\displaystyle\int\frac{1}{x^4+x^2+1}\,dx = \frac{1}{2}\int\left[\frac{(x^2+1)-(x^2-1)}{x^4+x^2+1}\right]\,dx\infty$

$$=\frac{1}{2}\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+(\sqrt{3})^2}-\frac{1}{2}\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-1} = \frac{1}{2}\int\frac{du}{u^2+(\sqrt{3})^2}-\frac{1}{2}\int\frac{dv}{v^2-1}$$

where जहाँ $u = x - 1/x \Rightarrow \left(1+\dfrac{1}{x^2}\right)dx = du$

and तथा $v = x + \Rightarrow \left(1-\dfrac{1}{x^2}\right)dx = dv$

$$I=\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)-\frac{1}{4}\ell n\left|\frac{v-1}{v+1}\right|+C$$

$$=\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right)-\frac{1}{4}\ell n\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right|+C$$

(ii) $I=\displaystyle\int\frac{1+x^2}{1+x^4}\,dx=\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+2-2}\,dx$

$$=\int\frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)+2}\,dx$$ Put $x-\dfrac{1}{x}=t$ रखने पर $\Rightarrow \left(1+\dfrac{1}{x^2}\right)dx = dt$

$$\Rightarrow I=\int\frac{dt}{t^2+2}=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+C=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\ x}\right)+C$$

(iii) $I=\displaystyle\int\frac{1-x^2}{1-x^2+x^4}\,dx=\int\frac{\frac{1}{x^2}-1}{\frac{1}{x^2}-1+x^2+2-2}\,dx$

$$= -\int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2 - \left(\sqrt{3}\right)^2}\,dx \qquad \text{Put } x + \frac{1}{x} = t \text{ रखने पर} \quad \Rightarrow \quad 1 - \frac{1}{x^2} = dt$$

$$\Rightarrow \qquad I = -\int \frac{dt}{t^2 - \left(\sqrt{3}\right)^2} = -\frac{1}{2\sqrt{3}} \log\left|\left(\frac{x+\frac{1}{x}-\sqrt{3}}{x+\frac{1}{x}+\sqrt{3}}\right)\right| + C$$

D-4. Integrate with respect to x : **[16JM120474]**

निम्नलिखित का x के सापेक्ष समाकलन कीजिए –

(i) $\dfrac{1}{(x+1)\sqrt{x+2}}$ (ii) $\dfrac{1}{(x^2-4)\sqrt{x+1}}$

(iii)✍ $\dfrac{1}{(x+1)\sqrt{x^2+2}}$ (iv)✍ $\dfrac{1}{(x^2+1)\sqrt{x^2+2}}$

Ans. (i) $\ell n \left| \frac{\sqrt{x+2}-1}{\sqrt{x+2}+1} \right| + C$ (ii) $\frac{1}{4\sqrt{3}} \ \ell n \left| \frac{t-\sqrt{3}}{t+\sqrt{3}} \right| - \frac{1}{2} \tan^{-1}(t) + C$, where जहाँ $t = \sqrt{x+1}$

(iii) $-\frac{1}{\sqrt{3}} \ell n \left| \left(t-\frac{1}{3}\right) + \sqrt{\left(t-\frac{1}{3}\right)^2 + \frac{2}{9}} \right| + C,$ where जहाँ $t = \frac{1}{x+1}$

(iv) $-\tan^{-1}\sqrt{\frac{x^2+2}{x^2}} + C$

Sol. **(i)** $I = \int \frac{dx}{(x+1)\sqrt{x+2}}$ Put $x + 2 = t^2$ रखने पर $\Rightarrow$ $dx = 2t\,dt$

$\Rightarrow$ $I = \int \frac{2t \ dt}{(t^2-1) \ . \ t} = 2. \ \frac{1}{2} \ \ell n \left|\frac{t-1}{t+1}\right| + C = \ell n \left| \frac{\sqrt{x+2}-1}{\sqrt{x+2}+1} \right| + C$

(ii) $I = \int \frac{1}{(x^2-4)\sqrt{x+1}} \ dx$ put $x + 1 = t^2$ रखने पर $dx = 2t\,dt$

$\Rightarrow$ $I = 2\int \frac{dt}{(t^2-3)(t^2+1)}$ (By partial fraction आंशिक भिन्न से)

$= \frac{1}{2}\int \frac{1}{t^2-\left(\sqrt{3}\right)^2} \ dt - \frac{1}{2}\int \frac{1}{t^2+1^2} \ dt = \frac{1}{4\sqrt{3}} \ \ell n \left| \frac{t-\sqrt{3}}{t+\sqrt{3}} \right| - \frac{1}{2}\tan^{-1}t + C,$ where जहाँ $t = \sqrt{x+1}$

(iii) $I = \int \frac{dx}{(x+1)\sqrt{x^2+2}}$ Put $x + 1 = \frac{1}{t}$ रखने पर $\Rightarrow dx = -\frac{1}{t^2} \ dt$

$\Rightarrow I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2+2}} = -\int \frac{dt}{t\sqrt{\frac{1}{t^2}-\frac{2}{t}+3}} = -\int \frac{dt}{\sqrt{3t^2-2t+1}} = -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t-\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}}$

$= -\frac{1}{\sqrt{3}} \ \ell n \left[\left(t-\frac{1}{3}\right) + \sqrt{\left(t-\frac{1}{3}\right)^2+\frac{2}{9}} \right] + C$ where जहाँ $t = \left(\frac{1}{x+1}\right)$

(iv) $I = \int \frac{dx}{\left(x^2+1\right) \ \sqrt{x^2+2}}$ Put $x = \frac{1}{t}$ रखने पर $\Rightarrow$ $dx = -\frac{1}{t^2} \ dt$

$\Rightarrow I = \int \frac{-\frac{1}{t^2}dt}{\left(\frac{1+t^2}{t^2}\right)\frac{\sqrt{1+2t^2}}{t}} = -\int \frac{t \ dt}{\left(1+t^2\right) \ \sqrt{1+2t^2}}$ Again put $1 + 2t^2 = z^2$ रखने पर $\Rightarrow 4t\,dt = 2z\,dz$

$\Rightarrow I = -\frac{1}{2}\int \frac{z \ dz}{\left(1+\frac{z^2-1}{2}\right) \ . \ z} = -\frac{2}{2}\int \frac{dz}{1+z^2} = -\tan^{-1}(z) + C = \tan^{-1}\sqrt{\left(\frac{x^2+2}{x^2}\right)} + C$

D-5. Evaluate the following :

निम्नलिखित को हल कीजिए–

(i) $\int \sqrt{\left(\frac{1+x}{x}\right)} dx$ (ii) $\int \sqrt{\left(\frac{x-1}{x+1}\right)} dx$ (iii) $\int \left(\frac{x\sqrt{1+x}}{\sqrt{1-x}}\right) dx$

Ans. (i) $\frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right| + \sqrt{x^2+x} + C$

(ii) $\sqrt{x^2-1} - \ln\left|x+\sqrt{x^2-1}\right| + C$ (iii) $\frac{1}{2}\sin^{-1}x - \frac{x}{2}\sqrt{1-x^2} - \sqrt{1-x^2} + C$

Sol. **(i)** $I = \int \frac{(1+x)}{\sqrt{x(x+1)}}dx = \frac{1}{2}\int\frac{(2x+2)}{\sqrt{x^2+x}}dx = \frac{1}{2}\int\frac{dx}{\sqrt{x^2+x}} + \frac{1}{2}\int\frac{(2x+1)}{\sqrt{x^2+x}}$

$= \frac{1}{2}\int\frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}} + \frac{1}{2}\int\frac{(2x+1)dx}{\sqrt{x^2+x}}$

$= \frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right| + \sqrt{x^2+x} + C$

(ii) $I = \int\left(\frac{x-1}{\sqrt{x^2-1}}\right)dx = \frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx - \int\frac{dx}{\sqrt{x^2-1}} = \sqrt{x^2-1} - \ln\left|x+\sqrt{x^2-1}\right| + C$

(iii) $I = \int\frac{(x(1+x))}{\sqrt{1-x^2}}dx = \int\frac{((x^2-1)+(x+1))}{\sqrt{1-x^2}}dx = -\int\sqrt{1-x^2}\,dx + \frac{1}{2}\int\frac{2x+2}{\sqrt{1-x^2}}dx$

$-\int\sqrt{1-x^2}\,dx - \frac{1}{2}\int\frac{-2xdx}{\sqrt{1-x^2}} + \int\frac{dx}{\sqrt{1-x^2}}$

$= -\left(\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right) - \sqrt{1-x^2} + \sin^{-1}x + C = \frac{1}{2}\sin^{-1}x - \frac{x}{2}\sqrt{1-x^2} - \sqrt{1-x^2} + C$

Section (E) : Integration of trigonometric functions :
खण्ड (E) : त्रिकोणमितीय फलनों का समाकलन :

E-1. Integrate with respect to x :
निम्नलिखित का x के सापेक्ष समाकलन कीजिए–

(i) $\frac{1}{2+\cos x}$ (ii) $\frac{1}{2-\cos x}$ (iii) $\frac{2\sin x+2\cos x}{3\cos x+2\sin x}$

(iv) $\frac{1}{1+\sin x+\cos x}$ (v) $\frac{1}{2+\sin^2 x}$ (vi) $\frac{\text{cosec}^2x.\sin x}{(\sin x - \cos x)}$ (vii) $\frac{\sin^4 x}{\cos^2 x}$

Ans. (i) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$ (ii) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\ \tan\frac{x}{2}\right) + C$

(iii) $\frac{10}{13}x - \frac{2}{13}\ell n\,|3\cos x + 2\sin x| + C$ (iv) $\ell n\left|1+\tan\frac{x}{2}\right| + C$

(v) $\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right) + C$ (vi) $\ell n\,|1-\cot x| + C$

(vii) $\tan x + \frac{1}{4}\sin 2x - \frac{3x}{2} + C$

Sol. **(i)** $I = \int\frac{dx}{2+\cos x}$

$= \int \frac{\sec^2\frac{x}{2}dx}{2\left(1+\tan^2\frac{x}{2}\right)+\left(1-\tan^2\frac{x}{2}\right)} = \int \frac{\sec^2\frac{x}{2}\,dx}{3+\tan^2\frac{x}{2}}$ put $\tan\frac{x}{2} = t$ रखने पर

$\Rightarrow \quad \frac{1}{2}\sec^2\frac{x}{2}dx = dt \quad \Rightarrow \quad I = \int \frac{2\ dt}{3+t^2} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + C$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + C$

(ii) $I = \int \frac{dx}{2-\cos x}$

$= \int \frac{\sec^2\frac{x}{2}dx}{2+2\tan^2\frac{x}{2}-1+\tan^2\frac{x}{2}}$

$= \int \frac{\sec^2\frac{x}{2}\,dx}{1+3\tan^2\frac{x}{2}}$ put $\sqrt{3}\tan\frac{x}{2} = t$ रखने पर $\Rightarrow \quad \frac{\sqrt{3}}{2}\sec^2\frac{x}{2}dx = dt$

$\Rightarrow \quad I = \frac{2}{\sqrt{3}}\int \frac{dt}{1+t^2}$

$= \frac{2}{\sqrt{3}}\tan^{-1}(t) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{x}{2}\right) + C$

(iii) $I = \int \frac{2\sin x+2\cos x}{3\cos x+2\sin x}dx$

put $2 \sin x + 2 \cos x = A(3 \cos x + 2 \sin x) + B(-3 \sin x + 2 \cos x)$ रखने पर

$\Rightarrow \quad 2 = 3A + 2B$ and तथा $2 = 2A - 3B$

$\Rightarrow \quad A = \frac{10}{13} \qquad B = -\frac{2}{13}$

$\Rightarrow \quad I = \int \frac{2\sin x+2\cos x}{3\cos x+2\sin x} = \frac{10}{13}\int dx - \frac{12}{13}\int \frac{dt}{t}$

$= \frac{10}{13}(x) - \frac{2}{13}\ln|(3\cos x + 2\sin x)| + C$

(iv) $I = \int \frac{1}{1+\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}dx$

$= \int \frac{\sec^2(x/2)}{2+2\tan x/2}dx$ put $\tan x/2 = t$ रखने पर $\Rightarrow \frac{1}{2}\sec^2(x/2)\,dx = dt$

$I = \int \frac{dt}{t+1} = \ell n\,|t + 1| + C = \ell n\left|1+\tan\frac{x}{2}\right| + C$

(v) $I = \int \frac{dx}{2+\sin^2 x} = \int \frac{\sec^2 x}{2\sec^2 x+\tan^2 x}\,dx$ put $\tan x = t$ रखने पर $\sec^2 x\,dx = dt$

$I = \int \frac{dt}{3t^2+2} = \frac{1}{3}\int \frac{dt}{t^2+(\sqrt{2/3})^2} = \frac{1}{3}\frac{\sqrt{3}}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2/3}}\right) + C = \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right) + C$

(vi) $I = \int \frac{\operatorname{cosec}^2 x.\sin x}{(\sin x-\cos x)}dx = \int \frac{\operatorname{cosec}^2 x\ \ dx}{(1-\cot x)}$ put $1 - \cot x = t$ रखने पर $\Rightarrow \operatorname{cosec}^2 x\,dx = dt$

$\Rightarrow \quad I = \ell n\,|t| + C = \ell n\,|(1 - \cot x)| + C$

(vii) $\int \frac{\sin^4 x}{\cos^2 x} dx = \int \frac{(1-\cos^2 x)^2}{\cos^2 x} dx = \int \frac{1+\cos^4 x - 2\cos^2 x}{\cos^2 x} dx$

$= \int \left(\sec^2 x + \frac{1+\cos 2x}{2} - 2 \right) dx = \int \sec^2 x \; dx + \frac{1}{2}\int \cos 2x \; dx - \frac{3}{2}\int dx$

$= \tan x + \frac{1}{4}\sin 2x - \frac{3}{2}x + C$

E-2. Evaluate the following

निम्न को हल कीजिए–

(i) $\int \left(\frac{\sin x + \cos x}{9 + 16\sin 2x} \right) dx$ (ii) $\int \left(\frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} \right) dx$

Ans. (i) $\frac{1}{40}\ell n\left(\frac{4(\sin x - \cos x) + 5}{4(\sin x + \cos x) - 5} \right) + C$ (ii) $\sin^{-1}\left(\frac{\sin x + \cos x}{3} \right) + C$

Sol. (i) $\int \frac{(\sin x + \cos x)dx}{9 - 16((\sin x - \cos x)^2 - 1)} = \int \frac{(\sin x + \cos x)dx}{25 - 16(\sin x - \cos x)^2}$

$= \int \frac{dt}{25 - 16t^2}$ {by putting रखने पर sinx – cosx = t}

$= \int \frac{dt}{25 - (4t)^2} = \frac{\frac{1}{10}\ell n\left(\frac{4t+5}{4t-5}\right)}{4} + C = \frac{1}{40}\ell n\left(\frac{4(\sin x - \cos x) + 5}{4(\sin x + \cos x) - 5} \right) + C$

(ii) $\int \left(\frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} \right) dx = \int \left(\frac{\cos x - \sin x}{\sqrt{8 - [(\sin x + \cos x)^2 - 1]}} \right) dx = \int \left(\frac{\cos x - \sin x}{\sqrt{9 - [(\sin x + \cos x)^2]}} \right) dx$

$= \frac{dt}{\sqrt{3^2 - t^2}} + C$ {sinx + cosx = t} $= \sin^{-1}\left(\frac{t}{3}\right) + C = \sin^{-1}\left(\frac{\sin x + \cos x}{3} \right) + C$

E-3. If $\int \sqrt{\frac{\cos^3 x}{\sin^{11} x}} \, dx = -2\left(A\tan^{\frac{-9}{2}} x + B\tan^{\frac{-5}{2}} x \right) + C$, then find A and B.

यदि $\int \sqrt{\frac{\cos^3 x}{\sin^{11} x}} \, dx = -2\left(A\tan^{\frac{-9}{2}} x + B\tan^{\frac{-5}{2}} x \right) + C$, हो, तो A और B का मान ज्ञात कीजिए।

Ans. $A = \frac{1}{9}$, $B = \frac{1}{5}$

Sol. $I = \int \sqrt{\frac{1}{(\tan x)^{11}(\cos^8 x)}} \; dx = \int (\tan x)^{-11/2} \sec^4 x dx$

$= \int (\tan x)^{-11/2}(1 + \tan^2 x)\sec^2 x dx$ put tanx = t रखने पर $\sec^2 x dx = dt$

$I = \int (t^{-11/2} + t^{-7/2}) \; dt$

$= \frac{-2}{9}t^{-9/2} + \frac{-2}{5}t^{-5/2} + C = \frac{-2}{9}(\tan x)^{-9/2} + \frac{-2}{5}(\tan x)^{-5/2} + C \quad \Rightarrow \quad A = \frac{1}{9}\ , B = \frac{1}{5}$

Section (F) : Reduction formulae
खण्ड (F) : समानयन सूत्र :

F-1. If $I_n = \int \frac{1}{(x^2+a^2)^n}dx$ then prove that $I_n = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\ I_{n-1}$

यदि $I_n = \int \frac{1}{(x^2+a^2)^n}dx$ तब सिद्ध कीजिए $I_n = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\ I_{n-1}$

Sol. $I_n = \int 1.(x^2+a^2)^{-n}dx \quad \Rightarrow \quad I_n = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}dx$

$\Rightarrow \quad I_n = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{((x^2+a^2)-a^2)}{(x^2+a^2)^{n+1}}dx \quad \Rightarrow \quad I_n = \frac{x}{(x^2+a^2)^n} + 2nI_n - 2a^2n\ I_{n+1}$

$\Rightarrow \quad 2a^2n\ I_{n+1} = (2n-1)I_n + \frac{x}{(x^2+a^2)^n} \quad \Rightarrow \quad I_n = \frac{(2n-3)I_{n-1}}{2a^2(n-1)} + \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}}$

F-2. If $I_n = \int x^n\ (a-x)^{1/2}dx$ then prove that $I_n = \frac{2an}{2n+3}\ I_{n-1} - \frac{2x^n(a-x)^{3/2}}{2n+3}$

यदि $I_n = \int x^n\ (a-x)^{1/2}dx$ तब सिद्ध कीजिए $I_n = \frac{2an}{2n+3}\ I_{n-1} - \frac{2x^n(a-x)^{3/2}}{2n+3}$

Sol. $I_n = \int x^n(a-x)^{1/2}dx \quad \Rightarrow \quad I_n = \frac{-2}{3}x^n(a-x)^{3/2} + \frac{2n}{3}\int x^{n-1}(a-x)^{3/2}dx$

$\Rightarrow \quad I_n = \frac{-2}{3}x^n(a-x)^{3/2} + \frac{2n}{3}\int x^{n-1}(a-x)(a-x)^{1/2}dx$

$\Rightarrow \quad I_n = \frac{-2}{3}x^n(a-x)^{3/2} + \frac{2an}{3}\int x^{n-1}(a-x)^{1/2}dx - \frac{2n}{3}\int x^n(a-x)^{1/2}dx$

$\Rightarrow \quad I_n + \frac{2n}{3}\ I_n = \frac{-2}{3}x^n\ (a-x)^{3/2} + \frac{2an}{3}I_{n-1} \quad \Rightarrow \quad I_n = \frac{2an}{2n+3}\ I_{n-1} - \frac{2x^n(a-x)^{3/2}}{2n+3}$

PART - II : ONLY ONE OPTION CORRECT TYPE

भाग - II : केवल एक सही विकल्प प्रकार (ONLY ONE OPTION CORRECT TYPE)

*** In each question C is arbitrary constant**
प्रत्येक प्रश्न में C स्वेच्छ अचर है।

Section (A) : Integration using Standard Integral :
खण्ड (A) : मानक समाकल की सहायता से समाकलन :

A-1. Integrate with respect to x : $\sqrt{x+1}$

$\sqrt{x+1}$ के सापेक्ष समाकलन कीजिए

(A) $\frac{(x+1)^{3/2}}{2}+C$ (B) $\frac{3(x+1)^{3/2}}{2}+C$ (C) $\frac{(x+1)^{3/2}}{3}+C$ (D*) $\frac{2(x+1)^{3/2}}{3}+C$

Sol. $I = \int\sqrt{x+1}\ dx = \int\sqrt{t}\ dt$ where जहाँ $x + 1 = t \Rightarrow dx = dt$

$\Rightarrow \quad I = \frac{2}{3}t^{3/2} = \frac{2}{3}(x+1)^{3/2} + C$

A-2 Integrate with respect to x : $\frac{1}{\sqrt{2x+1}}$

$\frac{1}{\sqrt{2x+1}}$ **का x के सापेक्ष समाकलन है–**

(A*) $\sqrt{2x+1}+C$ (B) $(2x+1)^{3/2}+C$ (C) $-\sqrt{2x+1}+C$ (D) $\frac{1}{(2x+1)^{3/2}}+C$

Sol. $I = \int\frac{1}{\sqrt{2x+1}}\ dx$ put $2x + 1 = t \Rightarrow dx = \frac{dt}{2}$ रखने पर

$\Rightarrow \quad I = \frac{1}{2}\int\frac{dt}{\sqrt{t}} = \sqrt{2x+1} + C$

A-3. If $\int\frac{1}{1+\sin x}\ dx = \tan\left(\frac{x}{2}+a\right) + C$, then **[16JM120475]**

(A*) $a = -\frac{\pi}{4}$, $C \in R$ (B) $a = \frac{\pi}{4}$, $C \in R$ (C) $a = \frac{5\pi}{4}$, $C \in R$ (D) $a = \frac{\pi}{3}$, $C \in R$

यदि $\int\frac{1}{1+\sin x}\ dx = \tan\left(\frac{x}{2}+a\right) + C$ हो, तो–

(A*) $a = -\frac{\pi}{4}$, $C \in R$ (B) $a = \frac{\pi}{4}$, $C \in R$ (C) $a = \frac{5\pi}{4}$, $C \in R$ (D) $a = \frac{\pi}{3}$, $C \in R$

Sol. $I = \int\frac{1}{1+\cos\left(\frac{\pi}{2}-x\right)}dx = \int\frac{1}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}dx = \frac{1}{2}\int\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)\ dx$

$= \tan\left(\frac{x}{2}-\frac{\pi}{4}\right)+C \Rightarrow a = -\frac{\pi}{4}, C \in R$

A-4. If $\int(\sin 2x - \cos 2x)\ dx = \frac{1}{\sqrt{2}}\sin(2x - a) + C$, then

यदि $\int(\sin 2x - \cos 2x)\,dx = \frac{1}{\sqrt{2}}\sin(2x - a) + C$ हो, तो–

(A) $a = \frac{5\pi}{4}$, $C \in R$ (B*) $a = -\frac{5\pi}{4}$, $C \in R$ (C) $a = \frac{\pi}{4}$, $C \in R$ (D) $a = \frac{\pi}{2}$, $C \in R$

Sol. $I = -\frac{1}{2}\cos 2x - \frac{\sin 2x}{2} + C = -\frac{1}{\sqrt{2}}\sin\left(2x+\frac{\pi}{4}\right)+C$

$= \frac{1}{\sqrt{2}}\sin\left(2x+\frac{5\pi}{4}\right)+C \Rightarrow a = -\frac{5\pi}{4},\ C \in R$

A-5. The value of $\int\frac{\cos 2x}{\cos x}\ dx$ is equal to **[16JM120476]**

(A*) $2 \sin x - \ell n |\sec x + \tan x| + C$ (B) $2 \sin x - \ell n |\sec x - \tan x| + C$
(C) $2 \sin x + \ell n |\sec x + \tan x| + C$ (D) $\sin x - \ell n |\sec x - \tan x| + C$

$\int \frac{\cos 2x}{\cos x}$ dx का मान है–
(A*) $2 \sin x - \ell n |\sec x + \tan x| + C$ (B) $2 \sin x - \ell n |\sec x - \tan x| + C$
(C) $2 \sin x + \ell n |\sec x + \tan x| + C$ (D) $\sin x - \ell n |\sec x - \tan x| + C$

Sol. $I = \int \frac{\cos 2x}{\cos x}\, dx = \int \frac{2\cos^2 x - 1}{\cos x}\, dx = 2 \sin x - \int \sec x\, dx = 2 \sin x - \ell n |\sec x + \tan x| + C$

A-6. If $\int \frac{\cos 4x + 1}{\cot x - \tan x}\, dx = A \cos 4x + B$; where A & B are constants, then
(A) $A = -1/4$ & B may have any value (B*) $A = -1/8$ & B may have any value
(C) $A = -1/2$ & $B = -1/4$ (D) $A = B = 1/2$

यदि $\int \frac{\cos 4x + 1}{\cot x - \tan x}\, dx = A \cos 4x + B$; जहाँ A एवं B अचर है, तब –
(A) $A = -1/4$ एवं B का कोई भी मान हो सकता है (B) $A = -1/8$ एवं B का कोई भी मान हो सकता है
(C) $A = -1/2$ एवं $B = -1/4$ (D) $A = B = 1/2$

Sol. $\int \frac{\cos 4x + 1}{\cot x - \tan x}\, dx = A \cos 4x + B$

L.H.S. वाम पक्ष $= \int \frac{2\cos^2 2x \sin x \cos x}{\cos 2x} dx = \int 2\cos 2x \sin x \cos x\, dx = \frac{1}{2}\int \sin 4x\, dx$

$= \frac{1}{2}\frac{(-\cos 4x)}{4} + B = -\frac{1}{8}\cos 4x + B$

Section (B) : Integration using Substitution :
खण्ड (B) : प्रतिस्थापन की सहायता से समाकलन :

B-1. The value of $\int \frac{a^{\sqrt{x}}}{\sqrt{x}}$ dx is equal to
(A) $\frac{a^{\sqrt{x}}}{\sqrt{x}} + C$ (B*) $\frac{2a^{\sqrt{x}}}{\ell n\ a} + C$ (C) $2a^{\sqrt{x}} . \ell n\, a + C$ (D) $2a^{\sqrt{x}} + C$

$\int \frac{a^{\sqrt{x}}}{\sqrt{x}}$ dx का मान है–
(A) $\frac{a^{\sqrt{x}}}{\sqrt{x}} + C$ (B*) $\frac{2a^{\sqrt{x}}}{\ell n\ a} + C$ (C) $2a^{\sqrt{x}} . \ell n\, a + C$ (D) $2a^{\sqrt{x}} + C$

Sol. $I = \int \frac{a^{\sqrt{x}}}{\sqrt{x}} dx$ Put $\sqrt{x} = t$ रखने पर $\Rightarrow$ $\frac{dx}{2\sqrt{x}} = dt$

$\Rightarrow$ $I = 2\int a^t\, dt = 2.\frac{a^{\sqrt{x}}}{\ell na} + C$

B-2. The value of $\int 5^{5^{5^x}} . 5^{5^x} . 5^x\, dx$ is equal to **[16JM120477]**

$\int 5^{5^{5^x}} . 5^{5^x} . 5^x\, dx$ का मान है–

(A) $\dfrac{5^{5^x}}{(\ell n\ 5)^3}+C$ (B) $5^{5^{5^x}}\ (\ell n\ 5)^3 + C$ (C*) $\dfrac{5^{5^{5^x}}}{(\ell n\ 5)^3}+C$ (D) $\dfrac{5^{5^{5^x}}}{(\ell n\ 5)^2}+C$

Sol. Let (माना) $5^{5^x} = t$

$\Rightarrow\ 5^{5^x}.\ \ \ell n\ 5\ .\ 5^x.\ \ \ell n\ 5\ dx = dt$

$$I = \int \frac{5^t dt}{(\ell n\ 5)^2} = \frac{5^t}{(\ell n\ 5)^3} + C = \frac{5^{5^{5^x}}}{(\ell n\ 5)^3} + C$$

B-3. The value of $\int \dfrac{\sqrt{\tan x}}{\sin x\ \cos x}\,dx$ is equal to

(A*) $2\sqrt{\tan x} + C$ (B) $2\sqrt{\cot x} + C$ (C) $\dfrac{\sqrt{\tan x}}{2} + C$ (D) $\sqrt{\tan x} + C$

$\int \dfrac{\sqrt{\tan x}}{\sin x\ \cos x}\,dx$ का मान है–

(A*) $2\sqrt{\tan x} + C$ (B) $2\sqrt{\cot x} + C$ (C) $\dfrac{\sqrt{\tan x}}{2} + C$ (D) $\sqrt{\tan x} + C$

Sol. $I = \int \dfrac{\sqrt{\tan x}}{\sin x\ \cos x}\,dx = \int \dfrac{\sqrt{\tan x}}{\tan x}\sec^2 x\ dx$ Put $\tan x = t$ रखने पर

$\Rightarrow\ \sec^2 x\, dx = dt$

$\Rightarrow\ I = \int \dfrac{1}{\sqrt{t}}dt = 2\sqrt{t} + C = 2\sqrt{\tan x} + C$

B-4. If $\int \dfrac{2^x}{\sqrt{1-4^x}}\,dx = K \sin^{-1}(2^x) + C$, then the value of K is equal to

यदि $\int \dfrac{2^x}{\sqrt{1-4^x}}\,dx = K \sin^{-1}(2^x) + C$ हो, तो K का मान है–

(A) $\ell n\ 2$ (B) $\dfrac{1}{2}\ell n\ 2$ (C) $\dfrac{1}{2}$ (D*) $\dfrac{1}{\ell n\ 2}$

Sol. Let (माना) $2^x = t \Rightarrow 2^x \log 2\, dx = dt$

$$I = \frac{1}{\ell n\ 2}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\ell n 2}\sin^{-1}(t) + C = \frac{1}{\ell n 2}\sin^{-1}(2^x) + C$$

$\therefore\ K = \dfrac{1}{\ell n\ 2}$

B-5. If $y = \int \dfrac{dx}{(1+x^2)^{3/2}}$ and y = 0 when x = 0, then value of y when x = 1, is:

यदि $y = \int \dfrac{dx}{(1+x^2)^{3/2}}$ और y = 0 हो जबकि x = 0, तो x = 1 पर y का मान है –

(A) $\sqrt{\dfrac{2}{3}}$ (B) $\sqrt{2}$ (C) $3\sqrt{2}$ (D*) $\dfrac{1}{\sqrt{2}}$

Sol. $y = \int \dfrac{dx}{x^3\left(1+\dfrac{1}{x^2}\right)^{3/2}}$ put $1 + \dfrac{1}{x^2} = t^2$ रखने पर $\Rightarrow\ -\dfrac{2}{x^3}dx = 2t\,dt$

$\Rightarrow \quad y = \int \frac{-t\ dt}{t^3} = -\int \frac{dt}{t^2} = +C = \frac{x}{\sqrt{1+x^2}} + C$

$\because \quad y(0) = 0 \quad \Rightarrow \quad C = 0 \quad \therefore \quad y(1) = \frac{1}{\sqrt{2}}$

B-6. The value of $\int \tan^3 2x \sec 2x\, dx$ is equal to : **[16JM120478]**

$\int \tan^3 2x \sec 2x\, dx$ का मान है –

(A) $\frac{1}{3}\sec^3 2x - \frac{1}{2}\sec 2x + C$ (B) $-\frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + C$

(C*) $\frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + C$ (D) $\frac{1}{3}\sec^3 2x + \frac{1}{2}\sec 2x + C$

Sol. $\int \tan^3 2x.\sec 2x\ \ dx$

$I = \int (\sec^2 2x - 1)\ \sec 2x.\tan 2x\ dx$ Put $\sec 2x = t$ रखने पर $\Rightarrow$ $2 \sec 2x \tan 2x\ dx = dt$

$\Rightarrow \quad I = \frac{1}{2}\int (t^2 - 1)\ dt = \frac{t^3}{6} - \frac{t}{2} + C = \frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + C$

B-7. If $\int x^{13/2} . (1+x^{5/2})^{1/2} dx = P(1+x^{5/2})^{7/2} + Q(1+x^{5/2})^{5/2} + R(1+x^{5/2})^{3/2} + C$, then P,Q and R are

यदि $\int x^{13/2} . (1+x^{5/2})^{1/2} dx = P(1+x^{5/2})^{7/2} + Q(1+x^{5/2})^{5/2} + R(1+x^{5/2})^{3/2} + C$, तब P , Q तथा R का मान है – **[16JM120480]**

(A*) $P = \frac{4}{35}$, $Q = -\frac{8}{25}$, $R = \frac{4}{15}$ (B) $P = \frac{4}{35}$, $Q = \frac{8}{25}$, $R = \frac{4}{15}$

(C) $P = -\frac{4}{35}$, $Q = -\frac{8}{25}$, $R = \frac{4}{15}$ (D) $P = \frac{4}{35}$, $Q = -\frac{8}{25}$, $R = -\frac{4}{15}$

Sol. Let माना $I = \int x^{13/2}(1+x^{5/2})^{1/2} dx = \int x^5.x^{3/2}(1+x^{5/2})^{1/2} dx$

Put $1 + x^{5/2} = t$ रखने पर $\Rightarrow x^{3/2} dx = dt$

$\Rightarrow I = \frac{2}{5}\int (t-1)^2 t^{1/2}\ dt = \frac{2}{5}\int (t^{5/2} + t^{1/2} - 2t^{3/2})\ dt = \frac{2}{5}\left[\frac{2}{7}t^{7/2} + \frac{2}{3}t^{3/2} - 2.\frac{2}{5}t^{5/2}\right] + C$

$= \frac{4}{35}(1+x^{5/2})^{7/2} + \frac{4}{15}(1+x^{5/2})^{3/2} - \frac{8}{25}(1+x^{5/2})^{3/2} + C$

On comparing with given, we have

तुलना करने पर

$P = \frac{4}{35}$, $Q = -\frac{8}{25}$, $R = \frac{4}{15}$

B-8. The value of $\int \frac{1-x^7}{x(1+x^7)} dx$ is equal to **[16JM120481]**

$\int \frac{1-x^7}{x(1+x^7)} dx$ का मान है–

(A) $\ell n|x| + \frac{2}{7}\ell n|1+x^7| + C$ (B) $\ell n|x| - \frac{2}{7}\ell n|1-x^7| + C$

(C*) $\ell n|x| - \frac{2}{7}\ell n|1+x^7| + C$ (D) $\ell n|x| + \frac{2}{7}\ell n|1-x^7| + C$

Sol. $\int \frac{1-x^7}{x\left(1+x^7\right)}\,dx = \int \frac{(1\;+\;x^7)-2x^7}{x\,(1\;+x^7)}dx \;=\; \int\frac{1}{x}dx - 2\int\frac{x^6}{(1+x^7)}dx \;= \ell n\,|x| - \frac{2}{7}\,\ell n\,|1+x^7| + C$

Section (C) : Integration by parts :
खण्ड (C) : खण्डशः समाकलन :

C-1. The value of $\int (x-1)\; e^{-x}\; dx$ is equal to

$\int (x-1)\; e^{-x}\; dx$ का मान है–

(A) $-xe^x + C$ (B) $xe^x + C$ (C*) $-xe^{-x} + C$ (D) $xe^{-x} + C$

Sol. $\int (x-1)\; e^{-x}\; dx$

$= (x-1)\int e^{-x}\;dx - \int\left(\frac{d}{dx}\;(x-1)\int e^{-x}dx\right)dx = -(x-1)\,e^{-x} - e^{-x} + C = -xe^{-x} + C$

C-2. The value of $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$ is equal to **[16JM120482]**

(A*) $x\,e^{\tan^{-1}x} + C$ (B) $x^2\; e^{\tan^{-1}x} + C$ (C) $\frac{1}{x}\; e^{\tan^{-1}x} + C$ (D) $x\,e^{\cot^{-1}x} + C$

$\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$ का मान है–

(A*) $x\,e^{\tan^{-1}x} + C$ (B) $x^2\; e^{\tan^{-1}x} + C$ (C) $\frac{1}{x}\; e^{\tan^{-1}x} + C$ (D) $x\; e^{\cot^{-1}x} + C$

Sol. Let (माना) $\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

$I = \int e^t(\sec^2 t + \tan t)\;\; dt = e^t \tan t + C$

$= \;x\,e^{\tan^{-1}x} + C$

C-3. The value of $\int [f(x)g''(x) - f''(x)g(x)]\; dx$ is equal to

$\int [f(x)g''(x) - f''(x)g(x)]\; dx$ का मान है–

(A) $\frac{f(x)}{g'(x)} + C$ (B) $f'(x)\,g(x) - f(x)\,g'(x) + C$

(C*) $f(x)\,g'(x) - f'(x)\,g(x) + C$ (D) $f(x)\,g'(x) + f'(x)\,g'(x) + C$

Sol. $\int (f(x)\;\; g''(x) - f''(x)g(x))\;\; dx \;=\; \int f(x)\;\; g''(x)\;\; dx \;-\; \int f''(x)\;\; g(x)\;\; dx$

$= f(x)\,g'(x) - \int f'(x)\;\; g'(x) \;-\; \int f''(x)\;\; g(x) = f(x)\,g'(x) - f'(x)\,g(x) + \int f''(x)\;\; g(x) - \int f''(x)\;\; g(x)$

$= f(x)\,g'(x) - f'(x)\,g(x) + C$

C-4. $\int \frac{x\,\ell n\,x}{\left(x^2-1\right)^{3/2}}\,dx$ equals **[16JM120483]**

$\int \frac{x\,\ell n\,x}{\left(x^2-1\right)^{3/2}}\;dx$ बराबर है–

(A*) $\text{arc}\sec x - \frac{\ell n\;x}{\sqrt{x^2-1}} + C$ (B) $\sec^{-1}x + \frac{\ell n\;x}{\sqrt{x^2-1}} + C$

(C) $\cos^{-1}x - \frac{\ell n\, x}{\sqrt{x^2-1}} + C$ (D) $\sec x - \frac{\ell n\, x}{\sqrt{x^2-1}} + C$

Sol. $I = \int \frac{x \ln x}{\left(x^2-1\right)^{3/2}}\, dx = \ell nx \int \frac{x\ dx}{(x^2-1)^{3/2}} - \int \left(\frac{1}{x} \cdot \int \frac{x}{(x^2-1)^{3/2}}\ dx\right) dx$ (Integration by parts)

(खण्डशः समाकलन से)

$= \ell nx \left(\frac{-1}{\sqrt{x^2-1}}\right) + \int \frac{dx}{x\sqrt{x^2-1}} = \frac{-\ell nx}{\sqrt{x^2-1}} + \sec^{-1}x + C$

C-5. The value of $\int (x\, e^{\ell n \sin x} - \cos x)\, dx$ is equal to:

$\int (x\, e^{\ell n \sin x} - \cos x)\, dx$ का मान है–

(A) $x \cos x + C$ (B) $\sin x - x \cos x + C$

(C*) $- e^{\ell n\, x} \cos x + C$ (D) $\sin x + x \cos x + C$

Sol. $\int \left(xe^{\ell n \sin x} - \cos x\right) dx = \int x \sin x - \int \cos x\ dx = -x \cos x + \int \cos x\ dx - \int \cos x\ dx$

$= - e^{\ell nx} \cos x + C$

Section (D) : Algebraic integral :
खण्ड (D) : बीजगणितीय समाकलन :

D-1. The value of $\int \frac{dx}{x^2+x+1}$ is equal to

(A) $\frac{\sqrt{3}}{2} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$ (B*) $\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

(C) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$ (D) $\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

$\int \frac{dx}{x^2+x+1}$ का मान है–

(A) $\frac{\sqrt{3}}{2} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$ (B*) $\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

(C) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$ (D) $\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

Sol. $I = \int \frac{dx}{(x+1/2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

D-2. The value of $\int \frac{1}{x^2(x^4+1)^{3/4}}\, dx$ is equal to **[16JM120484]**

$\int \frac{1}{x^2(x^4+1)^{3/4}}\, dx$ का मान है–

(A) $\left(1+\frac{1}{x^4}\right)^{1/4} + C$ (B) $(x^4+1)^{1/4} + C$ (C) $\left(1-\frac{1}{x^4}\right)^{1/4} + C$ (D*) $-\left(1+\frac{1}{x^4}\right)^{1/4} + C$

Sol. $I = \int \frac{1}{x^5(1+\frac{1}{x^4})^{3/4}} dx$ put $1+\frac{1}{x^4} = t$ रखने पर

$\Rightarrow -\frac{4}{x^5}dx = dt \quad I = -\frac{1}{4}\int\frac{dt}{t^{3/4}} = -\frac{1}{4}.\ \frac{t^{1/4}}{1/4}+C = -\left(1+\frac{1}{x^4}\right)^{1/4} + C$

Sol. $I = \int\frac{1}{x^5(1+\frac{1}{x^4})^{3/4}}\,dx$ put $1+\frac{1}{x^4} = t$ रखने पर $\Rightarrow -\frac{4}{x^5}dx = dt$

$\Rightarrow \quad I = -\frac{1}{4}\int\frac{dt}{t^{3/4}} = -\frac{1}{4}\ .\ \frac{t^{1/4}}{1/4}+C = -\left(1+\frac{1}{x^4}\right)^{1/4} + C$

D-3. The value of $\int\frac{dx}{x\sqrt{1-x^3}}$ is equal to

$\int\frac{dx}{x\sqrt{1-x^3}}$ का मान है–

(A*) $\frac{1}{3}\ \ell n\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + C$

(B) $\frac{1}{3}\ \ell n\left|\frac{\sqrt{1-x^2}+1}{\sqrt{1-x^2}-1}\right| + C$

(C) $\frac{1}{3}\ \ell n\left|\frac{1}{\sqrt{1-x^3}}\right| + C$

(D) $\frac{1}{3}\ \ell n\,|1-x^3| + C$

Sol. $I = \int\frac{dx}{x\sqrt{1-x^3}}$

$= \int\frac{x^2\ dx}{x^3\sqrt{1-x^3}}$ Put $1 - x^3 = t^2$ रखने पर $\Rightarrow$ $x^2\,dx = \frac{-2t\ \ dt}{3}$

$\Rightarrow \quad I = -\frac{2}{3}\int\frac{t.\ dt}{(1-t^2)\ .\ t} = \frac{2}{3}\int\frac{dt}{t^2-1} = \frac{2}{3}\times\frac{1}{2}\ \ell n\left|\frac{t-1}{t+1}\right| + C = \frac{1}{3}\ \ell n\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + C$

D-4. The value of $\int\sqrt{\frac{e^x-1}{e^x+1}}\ dx$ is equal to **[16JM120485]**

(A*) $\ell n\left(e^x+\sqrt{e^{2x}-1}\right) - \sec^{-1}(e^x) + C$

(B) $\ell n\left(e^x+\sqrt{e^{2x}-1}\right) + \sec^{-1}(e^x) + C$

(C) $\ell n\left(e^x-\sqrt{e^{2x}-1}\right) - \sec^{-1}(e^x) + C$

(D) $\ell n\left(e^x+\sqrt{e^{2x}-1}\right) - \sin^{-1}(e^x) + C$

$\int\sqrt{\frac{e^x-1}{e^x+1}}\ dx$ का मान है–

(A*) $\ell n\left(e^x+\sqrt{e^{2x}-1}\right) - \sec^{-1}(e^x) + C$

(B) $\ell n\left(e^x+\sqrt{e^{2x}-1}\right) + \sec^{-1}(e^x) + C$

(C) $\ell n\left(e^x-\sqrt{e^{2x}-1}\right) - \sec^{-1}(e^x) + C$

(D) $\ell n\left(e^x+\sqrt{e^{2x}-1}\right) - \sin^{-1}(e^x) + C$

Sol. $I = \int\frac{e^x-1}{\sqrt{e^{2x}-1}}\ dx$

$= \int\frac{e^x}{\sqrt{e^{2x}-1}}\ dx - \int\frac{e^x}{e^x\sqrt{e^{2x}-1}}dx$ put $e^x = t$ रखने पर $\Rightarrow$ $e^x\,dx = dt$

$\Rightarrow \quad I = \int\frac{dt}{\sqrt{t^2-1}} - \int\frac{dt}{t\sqrt{t^2-1}}$

$= \ell n\left|t+\sqrt{t^2-1}\right| - \sec^{-1}(t) + C,$ where (जहाँ) $t = e^x$

D-5. If $\int \frac{dx}{x^4+x^3} = \frac{A}{x^2} + \frac{B}{x} + \ell n\left|\frac{x}{x+1}\right| + C$, then

(A) $A = \frac{1}{2}$, $B = 1$ (B) $A = 1$, $B = -\frac{1}{2}$ (C*) $A = -\frac{1}{2}$, $B = 1$ (D) $A = -\frac{1}{2}$, $B = \frac{1}{2}$

यदि $\int \frac{dx}{x^4+x^3} = \frac{A}{x^2} + \frac{B}{x} + \ell n\left|\frac{x}{x+1}\right| + C$, हो, तो–

(A) $A = \frac{1}{2}$, $B = 1$ (B) $A = 1$, $B = -\frac{1}{2}$ (C*) $A = -\frac{1}{2}$, $B = 1$ (D) $A = -\frac{1}{2}$, $B = \frac{1}{2}$

Sol. $I = \int \frac{dx}{x^3(x+1)} = \frac{A}{x^2} + \frac{B}{x} + \ell n\left(\frac{x}{x+1}\right) + C$

let माना $\frac{1}{x^3(x+1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \frac{d}{(x+1)}$

$1 = a(x+1)x^2 + bx(x+1) + c(x+1)) + dx^3 = a\left(x^3+x^2\right) + b\left(x^2+x\right) + c(x+1) + dx^3$

$0 = a + d$

$0 = a + b$

$0 = b + c$

$c = 1, b = -1, a = 1, d = -1$

$\Rightarrow \int \frac{dx}{x^3(x+1)} = \int \frac{dx}{x} - \int \frac{dx}{x^2} + \int \frac{dx}{x^3} - \int \frac{1}{x+1}dx = \ell n|x| + \frac{1}{x} - \frac{1}{2x^2} - \ell n|x+1| + C$

Comparing तुलना करने पर

$B = 1, A = -\frac{1}{2}$

Section (E) : Integration of trigonometric functions :
खण्ड (E) : त्रिकोणमितीय फलनों का समाकलन :

E-1. The value of $\int \frac{\cos 2x}{(\sin x+\cos x)^2}$ dx is equal to

$\int \frac{\cos 2x}{(\sin x+\cos x)^2}$ dx का मान है–

(A) $\frac{-1}{\sin x+\cos x} + C$ (B*) ℓn (sin x + cos x) + C

(C) ℓn (sin x – cos x) + C (D) ℓn (sin x + cos x)^2 + C

Sol. $I = \int \frac{\cos 2x \ dx}{(\sin x+\cos x)^2} = \int \frac{\cos^2 x-\sin^2 x}{(\sin x+\cos x)^2} dx$

$= \int \frac{\cos x-\sin x}{\sin x+\cos x} dx$ Put sin x + cos x = t रखने पर $\Rightarrow$ (cos x – sin x) dx = dt

$\Rightarrow I = \int \frac{dt}{t} = \ell n\,|t| + C = \ell n|\sin x + \cos x| + C$

E-2 The value of $\int [1+\tan x \ . \ \tan(x+\alpha)]\,dx$ is equal to **[16JM120479]**

$\int [1+\tan x \ . \ \tan(x+\alpha)]$ dx का मान है–

(A) $\cos\alpha \,.\, \ell n \left|\dfrac{\sin x}{\sin(x+\alpha)}\right| + C$ (B) $\tan\alpha \,.\, \ell n \left|\dfrac{\sin x}{\sin(x+\alpha)}\right| + C$

(C*) $\cot\alpha \,.\, \ell n \left|\dfrac{\sec(x+\alpha)}{\sec x}\right| + C$ (D) $\cot\alpha \,.\, \ell n \left|\dfrac{\cos(x+\alpha)}{\cos x}\right| + C$

Sol. $\int (1+\tan x \ \tan(x+\alpha))\ dx = \int \dfrac{\sin x \ \sin(x+\alpha) + \cos x \cos(x+\alpha)}{\cos x \cos(x+\alpha)} = \int \dfrac{\cos(x+\alpha-x)}{\cos x \ \cos(x+\alpha)} dx$

$= \cot\alpha \int \dfrac{\sin(x+\alpha-x)}{\cos x \ \cos(x+\alpha)} dx = \cot\alpha \left[\int \dfrac{\sin(x+\alpha)\ \cos x}{\cos x \ \cos(x+\alpha)} - \dfrac{\cos(x+\alpha)\ \sin x}{\cos x \ \cos(x+\alpha)}\ dx\right]$

$= \cot\alpha \left[\int \tan(x+\alpha)\ dx - \int \tan x\ dx\right] = \cot\alpha\ [\ell n\,|\sec(x+\alpha)| - \ell n\,|\sec x|]$

$= \cot\alpha\ \ell n \left|\dfrac{\cos x}{\cos(x+\alpha)}\right| + C = \cot\alpha\ \ell n \left(\left|\dfrac{\sec(x+\alpha)}{\sec x}\right|\right) + C$

E-3. The value of $\int \sqrt{\sec x - 1}\ dx$ is equal to **[16JM120486]**

(A) $2\ \ell n \left(\cos\dfrac{x}{2} + \sqrt{\cos^2\dfrac{x}{2} - \dfrac{1}{2}}\right) + C$ (B) $\ell n \left(\cos\dfrac{x}{2} + \sqrt{\cos^2\dfrac{x}{2} - \dfrac{1}{2}}\right) + C$

(C*) $-2\ \ell n \left(\cos\dfrac{x}{2} + \sqrt{\cos^2\dfrac{x}{2} - \dfrac{1}{2}}\right) + C$ (D) $-2\ \ell n \left(\sin\dfrac{x}{2} + \sqrt{\cos^2\dfrac{x}{2} - \dfrac{1}{2}}\right) + C$

Sol. $I = \int \sqrt{\sec x - 1}\ dx = \int \dfrac{\sqrt{1-\cos x}}{\sqrt{\cos x + 1 - 1}}\ dx$

$= \int \dfrac{\sqrt{2}\sin\frac{x}{2}\ dx}{\sqrt{\left(\sqrt{2}\cos\frac{x}{2}\right)^2 - 1}}$ put $\sqrt{2}\ \cos = \dfrac{x}{2}$ t रखने पर $\Rightarrow$ $\sin\dfrac{x}{2}\ dx = -\sqrt{2}\ dt$

$\Rightarrow \quad I = -2\int \dfrac{dt}{\sqrt{t^2-1}} = -2\ell n\left(t + \sqrt{t^2-1}\right)$

$-2\ell n \left(\sqrt{2}\cos\dfrac{x}{2} + \sqrt{2\cos^2\dfrac{x}{2} - 1}\right) + C_1 = -2 \times \dfrac{1}{2}\ \ell n\, 2 - 2\ \ell n \left(\cos\dfrac{x}{2} + \sqrt{\cos^2\dfrac{x}{2} - \dfrac{1}{2}}\right) + C_1$

$= -2\ \ell n \left(\cos\dfrac{x}{2} + \sqrt{\cos^2\dfrac{x}{2} - \dfrac{1}{2}}\right) + C$

E-4. The value of $\int \dfrac{dx}{\cos^3 x\sqrt{\sin 2x}}$ is equal to

(A) $\sqrt{2}\left(\sqrt{\cos x} + \dfrac{1}{5}\tan^{5/2} x\right) + C$ (B*) $\sqrt{2}\left(\sqrt{\tan x} + \dfrac{1}{5}\tan^{5/2} x\right) + C$

(C) $\sqrt{2}\left(\sqrt{\tan x} - \dfrac{1}{5}\tan^{5/2} x\right) + C$ (D) $\sqrt{2}\left(\sqrt{\cos x} - \dfrac{1}{5}\tan^{5/2} x\right) + C$

$\int \dfrac{dx}{\cos^3 x\sqrt{\sin 2x}}$ का मान है–

(A) $\sqrt{2}\left(\sqrt{\cos x} + \dfrac{1}{5}\tan^{5/2} x\right) + C$ (B*) $\sqrt{2}\left(\sqrt{\tan x} + \dfrac{1}{5}\tan^{5/2} x\right) + C$

(C) $\sqrt{2}\left(\sqrt{\tan x}-\frac{1}{5}\tan^{5/2}x\right)+C$ (D) $\sqrt{2}\left(\sqrt{\cos x}-\frac{1}{5}\tan^{5/2}x\right)+C$

Sol. $I=\int\frac{dx}{\cos^3 x\sqrt{\sin 2x}}$

$=\int\frac{\sec^2 x.\sec^2 x\ \ dx}{\sqrt{2\tan x}}$ put $\tan x = t$ रखने पर $\Rightarrow$ $\sec^2 x\, dx = dt$

$\Rightarrow$ $I=\frac{1}{\sqrt{2}}\int\frac{1+t^2}{\sqrt{t}}\,dt=\frac{1}{\sqrt{2}}\int\left[t^{-\frac{1}{2}}+t^{\frac{3}{2}}\right]dt=\sqrt{2}\left(\sqrt{\tan x}+\frac{1}{5}(\tan x)^{5/2}\right)+C$

E-5. Antiderivative of $\frac{\sin^2 x}{1+\sin^2 x}$ w.r.t. x is : **[16JM120487]**

$\frac{\sin^2 x}{1+\sin^2 x}$ का x के सापेक्ष प्रति–अवकलज है –

(A*) $x-\frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}\tan x\right)+C$ (B) $x-\frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)+C$

(C) $x-\sqrt{2}\arctan\left(\sqrt{2}\tan x\right)+C$ (D) $x-\sqrt{2}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)+C$

Sol. $I=\int\frac{\sin^2 x+1-1}{1+\sin^2 x}dx=\int 1\ dx-\int\frac{1}{1+\sin^2 x}dx=x-\int\frac{\sec^2 x}{1+2\tan^2 x}dx=x-\frac{\sqrt{2}}{2}\tan^{-1}\left(\sqrt{2}\tan x\right)+C$

E-6. Integrate $\frac{1}{1-\cot x}$

समाकलन $\frac{1}{1-\cot x}$ कीजिए–

(A*) $\frac{1}{2}\log|\sin x-\cos x|+\frac{1}{2}x+C$ (B) $\frac{1}{2}\log|\sin x+\cos x|+\frac{1}{2}x+C$

(C) $\frac{1}{2}\log|\sin x+\cos x|-\frac{1}{2}x+C$ (D) $\frac{1}{2}\log|\sin x-\cos x|-\frac{1}{2}x+C$

Sol. Let माना $I=\int\frac{1}{1-\cot x}dx=\int\frac{\sin x\,dx}{\sin x-\cos x}$

Let माना $\sin x \equiv A(\cos x + \sin x) + B(\sin x - \cos x)$

then तब $\left.\begin{array}{l}A+B=1\\A-B=0\end{array}\right\}$ $\therefore$ $A=\frac{1}{2}$, $B=\frac{1}{2}$

$I=\int\frac{\frac{1}{2}(\cos x+\sin x)+\frac{1}{2}(\sin x-\cos x)}{(\sin x-\cos x)}dx=\frac{1}{2}\int\frac{\cos x+\sin x}{\sin x-\cos x}dx+\frac{1}{2}\int dx+C$

$=\frac{1}{2}\log|\sin x-\cos x|+\frac{1}{2}x+C$

E-7. $I=\int\frac{dx}{\sin x+\sec x}$ is equal to

(A*) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-(\sin x-\cos x)}\right|+\tan^{-1}(\sin x+\cos x)+C$

(B) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x - \cos x}{\sqrt{3}-(\sin x - \cos x)}\right| + \tan^{-1}(\sin x - \cos x) + C$

(C) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x + \cos x}{\sqrt{3}-(\sin x - \cos x)}\right| + \tan^{-1}(\sin x + \cos x) + C$

(D) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x - \cos x}{\sqrt{3}-(\sin x + \cos x)}\right| + \tan^{-1}(\sin x + \cos x) + C$

Sol. $I = \int \frac{\cos x}{1+\sin x\cos x}dx = \int \frac{2\cos x}{2+2\sin x\cos x} = \int \frac{(\cos x+\sin x)+(\cos x-\sin x)}{2+\sin 2x}$

$= \int \frac{d(\sin x-\cos x)}{2+(1-(\sin x-\cos x))^2} + \int \frac{d(\sin x+\cos x)}{2+((\sin x+\cos x)^2-1)}$

$= \int \frac{du}{(\sqrt{3})^2-u^2} + \int \frac{dv}{1+v^2}$ [where जहां u = sinx – cosx, v = sinx + cosx]

$= \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+u}{\sqrt{3}-u}\right| + \tan^{-1}v + C$

Section (F) : Reduction formulae
खण्ड (F) : समानयन सूत्र :

F-1. If $I_n = \int \frac{e^x}{x^n}dx$ and $I_n = \frac{-e^x}{k_1x^{n-1}} + \frac{1}{k_2-1}I_{n-1}$, then $(k_2 - k_1)$ is equal to :

(A) 0 (B*) 1 (C) 2 (D) 3

यदि $I_n = \int \frac{e^x}{x^n}dx$ और $I_n = \frac{-e^x}{k_1x^{n-1}} + \frac{1}{k_2-1}I_{n-1}$, तब $(k_2 - k_1)$ का मान बराबर होगा–

(A) 0 (B*) 1 (C) 2 (D) 3

Sol. $I_n = \int e^x\left(\frac{1}{x^n}\right)dx = \frac{e^x}{(1-n)x^{n-1}} - \int\left(\frac{1}{1-n}\right)\left(\frac{e^x}{x^{n-1}}\right)dx = \frac{-e^x}{(n-1)x^{n-1}} + \frac{1}{n-1}I_{n-1}$

F-2. If $I_n = \int \cot^n x \; dx$ and $I_0 + I_1 + 2(I_2 + ...+ I_8) + I_9 + I_{10} = A\left(u+\frac{u^2}{2}+.....+\frac{u^9}{9}\right) + C$, where u = cotx and C is an arbitrary constant, then

(A) A = 2 (B*) A = – 1 (C) A = 1 (D) A is dependent on x

यदि $I_n = \int \cot^n x \; dx$ और $I_0 + I_1 + 2(I_2 + ...+ I_8) + I_9 + I_{10} = A\left(u+\frac{u^2}{2}+.....+\frac{u^9}{9}\right) + C$, जहाँ u = cot x और C एक स्वेच्छ अचर है, तो–

(A) A = 2 (B*) A = – 1 (C) A = 1 (D) A, x पर निर्भर करता है

Sol. $I_n = \int \cot^n x \; dx = \int \cot^{n-2}x(\operatorname{cosec}^2 x - 1)\; dx = \frac{-\cot^{n-1}x}{n-1} - I_{n-2} + C_1$

$\Rightarrow \quad I_n + I_{n-2} = \frac{-\cot^{n-1}x}{n-1} + C_1$

Now, (अब) $I_0 + I_1 + 2(I_2 + ...+ I_8) + I_9 + I_{10}$

$= (I_2 + I_0) + (I_3 + I_1) + (I_4 + I_2) + (I_5 + I_3) + (I_6 + I_4) + (I_7 + I_5) + (I_8 + I_6) + (I_9 + I_7) + (I_{10} + I_8)$

$= -\left(\frac{\cot x}{1} + \frac{\cot^2 x}{2} + ... + \frac{\cot^9 x}{9}\right) + C$

$\therefore$ A = – 1.

PART - III : MATCH THE COLUMN

भाग - III : कॉलम को सुमेलित कीजिए (MATCH THE COLUMN)

1. **Column – I** **Column – II**

(A) If $F(x) = \int \frac{x + \sin x}{1 + \cos x}$ dx and F(0) = 0, then the value of F(π/2) is (p) $\frac{\pi}{2}$

(B) Let $F(x) = \int e^{\sin^{-1}x}\left(1 - \frac{x}{\sqrt{1-x^2}}\right)$ dx and F(0) = 1, (q) $\frac{\pi}{3}$

If $F(1/2) = \frac{k\sqrt{3}\ \ e^{\pi/6}}{\pi}$, then the value of k is

(C) Let $F(x) = \int \frac{dx}{(x^2+1)\ (x^2+9)}$ and F(0) = 0, (r) $\frac{\pi}{4}$

if $F(\sqrt{3}) = \frac{5}{36}k$, then the value of k is

(D) Let $F(x) = \int \frac{\sqrt{\tan x}}{\sin x \cos x}$ dx and F(0) = 0 (s) π

if $F(\pi/4) = \frac{2k}{\pi}$, then the value of k is

Ans. (A) → (p), (B) → (p), (C) → (r), (D) → (s)

स्तम्भ – I **स्तम्भ – II**

(A) यदि $F(x) = \int \frac{x + \sin x}{1 + \cos x}$ dx और F(0) = 0 हो, तो F(π/2) का मान है (p) $\frac{\pi}{2}$

(B) मानाकि $F(x) = \int e^{\sin^{-1}x}\left(1 - \frac{x}{\sqrt{1-x^2}}\right)$ dx और F(0) = 1 है, (q) $\frac{\pi}{3}$

यदि $F(1/2) = \frac{k\sqrt{3}\ \ e^{\pi/6}}{\pi}$ हो, तो k का मान है

(C) मानाकि $F(x) = \int \frac{dx}{(x^2+1)\ (x^2+9)}$ और F(0) = 0 है, (r) $\frac{\pi}{4}$

यदि $F(\sqrt{3}) = \frac{5}{36}$ k हो, तो k का मान है

(D) मानाकि $F(x) = \int \frac{\sqrt{\tan x}}{\sin x \cos x}$ dx और F(0) = 0 है, (s) π

यदि $F(\pi/4) = \frac{2k}{\pi}$ हो, तो k का मान है

Ans. (A) → (p), (B) → (p), (C) → (r), (D) → (s)

Sol. (A) $F(x) = \int \frac{x+\sin x}{1+\cos x}\, dx = \int \left(x \quad \frac{1}{2}\sec^2\frac{x}{2} \quad + \tan\frac{x}{2} \right) dx = x \tan x/2 + C$

Since (चूँकि) $0 = F(0)$ $\therefore$ $C = 0$ and (और) $F(\pi/2) = \pi/2$

(B) $F(x) = \int e^{\sin^{-1}x} \left(1 - \frac{x}{\sqrt{1-x^2}} \right) dx$

$= \int e^{\sin^{-1}x} \left(\frac{1}{\sqrt{1-x^2}} \quad \sqrt{1-x^2} + \frac{-x}{\sqrt{1-x^2}} \right) dx$ put $\sin^{-1}x = t$ रखने पर $\Rightarrow$ $\frac{dx}{\sqrt{1-x^2}}$ dt

$\Rightarrow$ $F(x) = \int e^t \left(\sqrt{1-\sin^2 t} - \sin t \right) dt = e^t \cos t + C = e^{\sin^{-1}x} \quad \sqrt{1-x^2} + C$

$\because$ $1 = F(0)$ $\Rightarrow$ $C = 0$

Hence, इस प्रकार $F(1/2) = e^{\pi/6}. \frac{\sqrt{3}}{2} = \frac{k\sqrt{3}}{\pi} e^{\pi/6}$ (given दिया गया है) $\therefore$ $k = \frac{\pi}{2}$

(C) $F(x) = \int \frac{dx}{(x^2+1) \quad (x^2+9)} = \frac{1}{8} \left(\frac{1}{x^2+1} - \frac{1}{x^2+9} \right) = \frac{1}{8} \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+9} \right) dx =$

$\frac{1}{8} \left[\tan^{-1} x - \frac{1}{3} \tan^{-1} \frac{x}{3} \right] + C$

$0 = F(0) = C$ $\therefore \frac{1}{8} \left(\frac{\pi}{3} - \frac{1}{3} . \frac{\pi}{6} \right) = \frac{5\pi}{144} = \frac{5k}{36}$ $\therefore$ $k = \frac{\pi}{4}$

(D) $F(x) = \int \frac{\sqrt{\tan x}}{\sin x \quad \cos x} dx = \int (\tan x)^{\frac{-1}{2}} \sec^2 x\, dx = 2 \sqrt{\tan x} + C$

$\because$ $0 = F(0)$ $\Rightarrow$ $C = 0$ $\therefore$ $F(\pi/4) = 2 = \frac{2k}{\pi}$ $\therefore$ $k = \pi$

2. If $I = \int \frac{dx}{a+b \quad \cos x}$, where a, b > 0 and a + b = u, a – b = v, then match the following column

यदि $I = \int \frac{dx}{a+b \quad \cos x}$, जहाँ a, b > 0 और a + b = u, a – b = v हो, तो निम्नलिखित स्तम्भों का मिलान कीजिए

Column – I **Column – II** **[16JM120488]**

स्तम्भ – I **स्तम्भ– II**

(A) v = 0 (p) $I = \frac{1}{\sqrt{uv}} \ell n \left| \frac{\sqrt{u}+\sqrt{v} \quad \tan\frac{x}{2}}{\sqrt{u}-\sqrt{v} \quad \tan\frac{x}{2}} \right| + C$

(B) v > 0 (q) $I = \frac{2}{\sqrt{uv}} \tan^{-1} \left(\sqrt{\frac{v}{u}} \quad \tan \frac{x}{2} \right) + C$

(C) v < 0 (r) $I = \frac{1}{\sqrt{-u \quad v}} \ell n \left| \frac{\sqrt{u}+\sqrt{-v} \quad \tan\frac{x}{2}}{\sqrt{u}-\sqrt{-v} \quad \tan\frac{x}{2}} \right|$

+ C

(s) $\frac{2}{u} \tan \frac{x}{2} + C$

Ans. (A) $\rightarrow$ (s) ; (B) $\rightarrow$ (q) ; (C) $\rightarrow$ (r)

Sol. (A) $v = 0 \Rightarrow a = b$, Also तथा $a + b = u \Rightarrow a = \frac{u}{2}$

Now, अब $I = \frac{1}{a}\int \frac{dx}{1+\cos x} = \frac{2}{u}\int \frac{dx}{2\cos^2\frac{x}{2}} = \frac{2}{u}\int \frac{1}{2}\sec^2\frac{x}{2}\ dx = \frac{2}{u}\ \tan\ \frac{x}{2} + C$

(B) $v > 0 \Rightarrow a > b$

Now, अब $I = \int \frac{dx}{a+b\cos x} = \int \frac{dx}{(a-b)+2b\cos^2\frac{x}{2}} = \int \frac{\sec^2\frac{x}{2}dx}{(a+b)+(a-b)\tan^2\frac{x}{2}}$

put $\tan\ \frac{x}{2} = t$ रखने पर $\Rightarrow\ \frac{1}{2}\ \sec^2\ \frac{x}{2}\ dx = dt$

$\Rightarrow I = \int \frac{2\ \ dt}{(a+b)+(a-b)+t^2} = \frac{2}{a-b}\int \frac{dt}{t^2+\left(\frac{a+b}{a-b}\right)}$ ...(1)

$= \frac{2}{a-b}\ \sqrt{\frac{a-b}{a+b}}\ \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\ \ t\right) + C = \frac{2}{\sqrt{uv}}\ \tan^{-1}\left(\sqrt{\frac{v}{u}}\tan\frac{x}{2}\right) + C$

(C) $v < 0 \Rightarrow a - b < 0 \Rightarrow b - a > 0$

Now अब $I = \frac{2}{b-a}\int \frac{dt}{\frac{a+b}{b-a}-t^2}$ (using equation (1) of part (B)) (भाग (B) की समीकरण (1) का उपयोग करने पर)

$= \frac{2}{b-a}\ \frac{1}{2}\sqrt{\frac{b-a}{b+a}}\ \ln\left|\frac{\sqrt{\frac{b+a}{b-a}}+t}{\sqrt{\frac{b+a}{b-a}}-t}\right| + C = \frac{1}{\sqrt{-uv}}\ \ln\left|\frac{\sqrt{u}+\sqrt{-v}\tan\frac{x}{2}}{\sqrt{u}-\sqrt{-v}\tan\frac{x}{2}}\right| + C$

Exercise-2

✍ **Marked questions are recommended for Revision.**
✍ **चिन्हित प्रश्न दोहराने योग्य प्रश्न है।**

PART - I : ONLY ONE OPTION CORRECT TYPE

भाग-I : केवल एक सही विकल्प प्रकार (ONLY ONE OPTION CORRECT TYPE)

*** In each question C is arbitrary constant**
प्रत्येक प्रश्न में C स्वेच्छ अचर है।

1.✍ Value of $\int \frac{1}{\sin(x-a)\cos(x-b)}\,dx$ is equal to

$\int \frac{1}{\sin(x-a)\cos(x-b)}\,dx$ बराबर है

(A*) $\frac{1}{\cos(a-b)}\ \ell n\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C$

(B) $\frac{1}{\cos(a-b)}\ \ell n\left|\frac{\cos(x-b)}{\sin(x-a)}\right| + C$

(C) $\frac{1}{\sin(a-b)}\ \ell n\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C$

(D) $\frac{1}{\sin(a+b)}\ \ell n\left|\frac{\cos(x-a)}{\sin(x-b)}\right| + C$

Sol. $\int \frac{dx}{\sin(x-a)\cos(x-b)}$

$$= \frac{1}{\cos(b-a)}\int \frac{\cos((x-a)-(x-b))}{\sin(x-a)\cos(x-b)}dx = \frac{1}{\cos(b-a)}\int\left[\frac{\cos(x-a)\cos(x-b)}{\sin(x-a)\cos(x-b)} + \frac{\sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}\right]dx$$

$$= \frac{1}{\cos(b-a)}\left[\int \cot(x-a)\ \ dx + \int \tan(x-b)\ \ dx\right]$$

$$= \frac{1}{\cos(a-b)}\ [\ell n\,|\sin(x-a)| - \ell n\,|\cos(x-b)|] + C = \frac{1}{\cos(a-b)}\ \ell n\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C$$

2. $\int$ tan x. tan 2x. tan 3x dx = **[16JM120489]**

(A) $-\ell n\,|\cos x| - \frac{1}{2}\ell n\,|\sec 2x| + \frac{1}{3}\ell n\,|\sec 3x| + C$

(B*) $-\ell n\,|\sec x| - \frac{1}{2}\ell n\,|\sec 2x| + \frac{1}{3}\ell n\,|\sec 3x| + C$

(C) $\ell n\,|\cos x| + \ell n\,|\cos 2x| + \ell n\,|\cos 3x| + C$

(D) $\ell n\,|\sec x| + \frac{1}{2}\ell n\,|\sec 2x| + \frac{1}{3}|\sec 3x| + C$

Sol. $\int \tan x\ \ \tan 2x\ \ \tan 3x\ \ dx$

$= \int(\tan 3x - \tan x - \tan 2x)\ dx$ as जैसे कि $\tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x\ .\ \tan x}$

$= -\frac{1}{3}\ell n\,|\cos 3x| + \ell n\,|\cos x| + \frac{1}{2}\ \ell n\,|\cos 2x| + C$

$= -\ell n\,|\sec x| + \frac{1}{3}\ \ell n|\sec 3x| - \frac{1}{2}\ \ell n\,|\sec 2x| + C$

3. The value of $\int$ (sin x. cos x. cos 2x. cos 4x. cos 8x. cos 16 x) dx is equal to

$\int$ (sin x. cos x. cos 2x. cos 4x. cos 8x. cos 16 x) dx का मान है –

(A) $\frac{\sin 16x}{1024} + C$ (B*) $-\frac{\cos 32x}{1024} + C$ (C) $\frac{\cos 32x}{1096} + C$ (D) $-\frac{\cos 32x}{1096} + C$

Sol. $\int \sin x.\cos x.\cos 2x.\cos 4x.\cos 8x.\cos 16x \quad dx$ =

$\frac{1}{2}\int 2\sin x \quad \cos x.\cos 2x \quad \cos 4x \quad \cos 8x \quad \cos 16x \quad dx$

$= \frac{1}{2^5}\int \sin 32 \quad x \ dx = -\frac{1}{2^5} \times \frac{1}{32} \times \cos 32x + C = -\frac{\cos 32x}{1024} + C$

4. $\int x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\, dx =$

(A) $\frac{1}{2}a^2\cos^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\sqrt{a^4 + x^4} + C$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right) + \sqrt{a^4 + x^4} + C$

(C*) $\frac{1}{2}a^2\sin^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\sqrt{a^4 - x^4} + C$ (D) $\frac{1}{2}\cos^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\sqrt{a^4 - x^4} + C$

Sol. $I = \int x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\, dx$ put $x^2 = a^2\cos 2\theta$ रखने पर $\Rightarrow 2x\,dx = -2a^2\sin 2\theta\, d\theta$

$\Rightarrow I = \int\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\,(-a^2\sin 2\theta)\,d\theta \quad = -a^2\int\frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta\, d\theta$

$= -a^2\int(1-\cos 2\theta)\ d\theta \ = -a^2\theta + \frac{a^2\sin 2\theta}{2} + C_1 = \frac{a^2}{2}\sqrt{1-\frac{x^4}{a^4}} - \frac{a^2}{2}\cos^{-1}\left(\frac{x^2}{a^2}\right) + C_1$

$= \frac{1}{2}\sqrt{a^4 - x^4} + \frac{a^2}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right) + C$

5. The value of $\int\sqrt{\frac{x-1}{x+1}}\cdot\frac{1}{x^2}\,dx$ is equal to **[16JM120490]**

$\int\sqrt{\frac{x-1}{x+1}}\cdot\frac{1}{x^2}\,dx$ का मान है–

(A) $\sin^{-1}\frac{1}{x} + \frac{\sqrt{x^2-1}}{x} + C$ (B) $\frac{\sqrt{x^2-1}}{x} + \cos^{-1}\frac{1}{x} + C$

(C*) $\sec^{-1}x - \frac{\sqrt{x^2-1}}{x} + C$ (D) $\tan^{-1}\sqrt{x^2+1} - \frac{\sqrt{x^2-1}}{x} + C$

Sol. $I = \int\sqrt{\frac{x-1}{x+1}} \times \frac{1}{x^2}\,dx$ Put $\frac{1}{x} = \cos 2\theta$ रखने पर $\Rightarrow -\frac{dx}{x^2} = -2\sin 2\theta\, d\theta$

$I = \int\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\ 2\sin 2\theta\, d\theta$

$= \int 4\sin^2\theta \quad d\theta \ = 2\int(1-\cos 2\theta)\ d\theta \ = 2\theta - \sin 2\theta + C = \cos^{-1}\left(\frac{1}{x}\right) - \sqrt{1-\frac{1}{x^2}} + C$

6. The value of $\int\frac{\ell n|x|}{x\sqrt{1+\ell n|x|}}\,dx$ equals :

$\int \frac{\ell n|x|}{x\sqrt{1+\ell n|x|}}$ dx का मान है–

(A*) $\frac{2}{3}\sqrt{1+\ell n|x|}\ (\ell n|x| - 2) + C$ (B) $\frac{2}{3}\sqrt{1+\ell n|x|}\ (\ell n|x| + 2) + C$

(C) $\frac{1}{3}\sqrt{1+\ell n|x|}\ (\ell n|x| - 2) + C$ (D) $2\sqrt{1+\ell n|x|}\ (3\ \ell n|x| - 2) + C$

Sol. $I = \int \frac{\ell n|x|}{x\sqrt{1+\ell n|x|}}\,dx$ Put $1 + \ell n|x| = t^2$, then $\frac{1}{x}\,dx = 2t\,dt$

$\Rightarrow$ $I = \int \frac{t^2-1}{t}.2t\,dt = 2\left[\frac{t^3}{3} - t\right] + C = \frac{2}{3}\sqrt{1+\ell n|x|}\ (\ell n|x| - 2) + C$

Hindi. $\int \frac{\ell n|x|}{x\sqrt{1+\ell n|x|}}\,dx$

मानाकि $1 + \ell n|x| = t^2$, तो $\frac{1}{x}\,dx = 2t\,dt$

$\Rightarrow$ $I = \int \frac{t^2-1}{t}.2t\,dt = 2\left[\frac{t^3}{3} - t\right] + C = \frac{2}{3}\sqrt{1+\ell n|x|}\ (\ell n|x| - 2) + C$

7. The value of $\int \frac{1}{[(x-1)^3(x+2)^5]^{1/4}}\,dx$ is equal to

$\int \frac{1}{[(x-1)^3(x+2)^5]^{1/4}}\,dx$ का मान है–

(A*) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$ (B) $\frac{4}{3}\left(\frac{x+2}{x-1}\right)^{1/4} + C$

(C) $\frac{1}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$ (D) $\frac{1}{3}\left(\frac{x+1}{x-1}\right)^{1/4} + C$

Sol. $I = \int \frac{1}{[(x-1)^3(x+2)^5]^{1/4}}\,dx$

$= \int \frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}\,dx$ put $\frac{x-1}{x+2} = t$ रखने पर $\Rightarrow$ $\frac{3}{(x+2)^2}\,dx = dt$

$\Rightarrow$ $I = \frac{1}{3}\int t^{-3/4}dt = \frac{1}{3}\ \frac{t^{1/4}}{(1/4)} + C = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$

8. The value of $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$ is equal to

$\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$ का मान है–

(A*) $\sqrt{x}\sqrt{1-x} - 2\sqrt{1-x} + \cos^{-1}\left(\sqrt{x}\right) + C$ (B) $\sqrt{x}\sqrt{1-x} + 2\sqrt{1-x} + \cos^{-1}\left(\sqrt{x}\right) + C$

(C) $\sqrt{x}\sqrt{1-x} - 2\sqrt{1-x} - \cos^{-1}\left(\sqrt{x}\right) + C$ (D) $\sqrt{x}\sqrt{1-x} + 2\sqrt{1-x} - \cos^{-1}\left(\sqrt{x}\right) + C$

Sol. $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$ put $\sqrt{x} = \cos 2t$ रखने पर $\Rightarrow$ $dx = -4\sin 2t\cos 2t\,dt$

$= \int \sqrt{\dfrac{1-\cos 2t}{1+\cos 2t}}\ (-4 \sin 2t \cos 2t)\ dt \ = -4\int \dfrac{\sin t}{\cos t}\ .2\sin t \cos t \cos 2t\ .dt$

$= -4\int (1-\cos 2t)\cos 2t\ dt \qquad = -4 \int \cos 2t \ \ dt \ + 4\int \cos^2 2t \ \ dt$

$= -\dfrac{4}{2} \sin\ 2t + 2\int (\cos 4t + 1)\ \ dt \qquad = -2 \sin 2t + 2 \times \dfrac{\sin 4t}{4} \ + 2t + C$

$= -2\ \sqrt{1-x}\ + \sqrt{x}\ \sqrt{1-x}\ + \cos^{-1} \sqrt{x}\ + C$

9. $\int \ \sin^{-1} \sqrt{\dfrac{x}{a+x}}\ dx$ is equal to **[16JM120491]**

(A*) $(a + x)$ arc tan $\sqrt{\dfrac{x}{a}}\ - \sqrt{ax}\ + C$

(B) $(a + x)$ arc tan $\sqrt{\dfrac{x}{a}} + \sqrt{ax}\ + C$

(C) $(a - x)$ arc tan $\sqrt{\dfrac{x}{a}}\ - \sqrt{ax}\ + C$

(D) $(a + x)$ arc cot $\sqrt{\dfrac{x}{a}}\ - \sqrt{ax}\ + C$

Sol. $I = \int \sin^{-1}\sqrt{\dfrac{x}{a+x}}\ dx$ put $x = a \tan^2\theta$ रखने पर $\Rightarrow$ $dx = 2a \tan\theta \sec^2\theta\, d\theta$

$\therefore \qquad I = \int \sin^{-1}\sqrt{\dfrac{a\tan^2\theta}{a\sec^2\theta}}\ \times 2a \tan\theta \sec^2\theta\ d\theta$

$= 2a \int \theta \ \ \tan\theta \sec^2\theta \ \ d\theta \ = 2a[\theta \ \ \int \tan\theta \ \ \sec^2\theta \ \ d\theta - \int (1 \ \ \int \tan\theta \ \ \sec^2\theta \ \ d\theta) \ \ d\theta]$

$= 2a\left[\theta \ \ \dfrac{\sec^2\theta}{2} - \dfrac{1}{2}\int \sec^2\theta \ \ d\theta\right]$

$= a\,[\theta \tan^2\theta - \tan\theta + \theta] + C = a\left[\dfrac{x}{a}\tan^{-1}\ \sqrt{\dfrac{x}{a}} - \sqrt{\dfrac{x}{a}} + \tan^{-1}\sqrt{\dfrac{x}{a}}\ \right] + C$

$= (a + x)\ \tan^{-1} \sqrt{\dfrac{x}{a}}\ - \sqrt{ax}\ + C$

10. The value of $\int \dfrac{e^{\sqrt{x}}}{\sqrt{x}}\left(x+\sqrt{x}\right) dx$ is equal to :

$\int \dfrac{e^{\sqrt{x}}}{\sqrt{x}}\left(x+\sqrt{x}\right)$ dx का मान है–

(A) $2e^{\sqrt{x}}\ [\sqrt{x} - x + 1\ ] + C$

(B) $2e^{\sqrt{x}}\ [x - 2\sqrt{x} + 1\ ] + C$

(C*) $2e^{\sqrt{x}}\ [x - \sqrt{x} + 1\ ] + C$

(D) $2e^{\sqrt{x}}\ (x + \sqrt{x} + 1\ ) + C$

Sol. $I = \int \dfrac{e^{\sqrt{x}}(x+\sqrt{x})}{\sqrt{x}}\ \ dx$ put $\sqrt{x}\ = t$ रखने पर $\Rightarrow$ $\dfrac{1}{\sqrt{x}}dx\ = 2\ dt$

$\Rightarrow \quad I \ = \ 2 \ \int e^t\left(t^2+t\right) \ dt \ = \ 2 \ \left[e^t.t^2 - \int 2t.e^t \ \ dt + \int e^t.t \ \ dt\right] \ = \ 2\left[e^t.t^2 - \int t.e^t \ \ dt\right] \ = \ 2$
$\left[e^t.t^2 - t.e^t + e^t\right] + C$

$= 2e^t\ \left[t^2 - t + 1\right] + C = 2\,e^{\sqrt{x}}\ [x - \ \sqrt{x}\ + 1] + C$

11. If $I = \int \dfrac{2}{x}(x^{\ell nx})(\ell nx)^3 dx = Ax^{\ell nx}(\ell nx)^2 - B\,x^{\ell nx} + C$, then $\dfrac{A}{B}$ is equal to :

(A*) 1 (B) –1 (C) 2 (D) –2

यदि $I = \int \frac{2}{x}(x^{\ell nx})(\ell nx)^3 dx = Ax^{\ell nx}(\ell nx)^2 - B\, x^{\ell nx} + C$, तब $\frac{A}{B}$ बराबर है–

(A*) 1 (B) –1 (C) 2 (D) –2

Sol. $I = \int \frac{2\ell nx}{x}(x^{\ell nx})(\ell n(x^{\ell nx}))dx$

put रखने पर $x^{\ell nx} = t \quad \Rightarrow \quad \frac{2\ell nx}{x}(x^{\ell nx}) = dt$

$I = \int \ell n t dt = t\ell nt - t + C = x^{\ell nx}(\ell nx)^2 - x^{\ell nx} + C$

12. The value of $\int e^{\tan\theta}(\sec\theta - \sin\theta)\, d\theta$ is equal to **[16JM120492]**

$\int e^{\tan\theta}(\sec\theta - \sin\theta)\, d\theta$ का मान है–

(A) $-e^{\tan\theta}\sin\theta + C$ (B) $e^{\tan\theta}\sin\theta + C$ (C) $e^{\tan\theta}\sec\theta + C$ (D*) $e^{\tan\theta}\cos\theta + C$

Sol. $I = \int e^{\tan\theta}(\sec\theta - \sin\theta)\, d\theta = \int e^{\tan\theta}\left(\frac{\sec\theta - \sin\theta}{\sec^2\theta}\right)\sec^2\theta\, d\theta$

$I = \int e^{\tan\theta}\left(\frac{1}{\sec\theta} - \frac{\sin\theta}{\sec^2\theta}\right)\sec^2\theta\, d\theta = \int e^{\tan\theta}\left(\frac{1}{\sqrt{1+\tan^2\theta}} - \frac{\tan\theta}{(1+\tan^2\theta)^{3/2}}\right)\sec^2\theta\, d\theta$

$= e^{\tan\theta} . \frac{1}{\sqrt{1+\tan^2\theta}} = e^{\tan\theta}. \cos\theta + C$

13. The value of $\int\left\{\ln(1+\sin x) + x\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\}dx$ is equal to:

$\int\left\{\ln(1+\sin x) + x\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\}dx$ का मान है–

(A*) $x\, \ell n\,(1 + \sin x) + C$ (B) $\ell n\,(1 + \sin x) + C$

(C) $-x\, \ell n\,(1 + \sin x) + C$ (D) $\ell n\,(1 - \sin x) + C$

Sol. $\int\left\{\ln(1+\sin x) + x\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\}dx = \int \ell n(1+\sin x)\, dx + \int x\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) dx$

$= x.\ell n\,(1+\sin x) - \int \frac{1}{1+\sin x}\cos x.\ x\ dx + \int x\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) dx$

$= x.\ell n\,(1+\sin x) - \int \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}.\ x\ dx + \int x\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) dx = x\, \ell n\,(1 + \sin x) + C$

14. The value of $\int x.\frac{\ell n\left(x + \sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$ equals: **[16JM120493]**

$\int x.\frac{\ell n\left(x + \sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$ का मान है–

(A*) $\sqrt{1+x^2}\ \ell n\left(x + \sqrt{1+x^2}\right) - x + C$

(B) $\frac{x}{2}.\ \ell n^2\left(x+\sqrt{1+x^2}\right)-\frac{x}{\sqrt{1+x^2}}+C$

(C) $\frac{x}{2}.\ \ell n^2\left(x+\sqrt{1+x^2}\right)+\frac{x}{\sqrt{1+x^2}}+C$

(D) $\sqrt{1+x^2}\ \ell n\left(x+\sqrt{1+x^2}\right)+x+C$

Sol. $\int x\ .\ \frac{\ell n(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx$

$I=\int \ell n\ (x+\sqrt{1+x^2})\ .\ \frac{x}{\sqrt{1+x^2}}\,dx$

$=\sqrt{1+x^2}\ \ell n\ (x+\sqrt{1+x^2})-\int\frac{1}{\sqrt{1+x^2}}.\ \sqrt{1+x^2}\ dx$ (using integration by parts)

$=\sqrt{1+x^2}\ \ell n\ (x+\sqrt{1+x^2})-x+C$

Alternate:

Let $x+\sqrt{1+x^2}=e^t$ then $\sqrt{1+x^2}-x=e^{-t}$

$\frac{dx}{\sqrt{1+x^2}}=dt$

$I=\int x\ t\ dt=\int\frac{(e^t-e^{-t})t\ dt}{2}=\frac{1}{2}t(e^t+e^{-t})-\frac{1}{2}(e^t-e^{-t})+C$

$=\sqrt{1+x^2}\ \ell n\ (x+\sqrt{1+x^2})-x+C.$

Hindi $\int x\ .\ \frac{\ell n(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx$

$I=\int \ell n\ (x+\sqrt{1+x^2})\ .\ \frac{x}{\sqrt{1+x^2}}\,dx$

$=\sqrt{1+x^2}\ \ell n\ (x+\sqrt{1+x^2})-\int\frac{1}{\sqrt{1+x^2}}.\ \sqrt{1+x^2}\ dx$ (खण्डशः समाकलन का प्रयोग करने पर)

$=\sqrt{1+x^2}\ \ell n\ (x+\sqrt{1+x^2})-x+C$

वैकल्पिक हल :

माना $x+\sqrt{1+x^2}=e^t$ तब $\sqrt{1+x^2}-x=e^{-t}$

$\frac{dx}{\sqrt{1+x^2}}=dt$

$I=\int x\ t\ dt=\int\frac{(e^t-e^{-t})t\ dt}{2}=\frac{1}{2}t(e^t+e^{-t})-\frac{1}{2}(e^t-e^{-t})+C$

$=\sqrt{1+x^2}\ \ell n\ (x+\sqrt{1+x^2})-x+C.$

15. If $\int\frac{x\tan^{-1}x}{\sqrt{1+x^2}}\,dx=\sqrt{1+x^2}\ f(x)+A\ \ell n\left|x+\sqrt{x^2+1}\right|+C$, then

यदि $\int\frac{x\tan^{-1}x}{\sqrt{1+x^2}}\,dx=\sqrt{1+x^2}\ f(x)+A\ \ell n\left|x+\sqrt{x^2+1}\right|+C$, तब

(A*) $f(x)=\tan^{-1}x,\ A=-1$
(B) $f(x)=\tan^{-1}x,\ A=1$
(C) $f(x)=2\tan^{-1}x,\ A=-1$
(D) $f(x)=2\tan^{-1}x,\ A=1$

Sol. $\int \frac{x\tan^{-1}x \quad dx}{\sqrt{1+x^2}}$ Put x = tan θ ⇒ dx = sec²θ dθ

⇒ I = $\int \frac{\tan\theta \;.\; \theta}{\sec\theta}\;.\sec^2\theta \quad d\theta \;= \theta \int \tan\theta\sec\theta \quad d\theta - \int 1 \;.\; \sec\theta \quad d\theta$ = θ sec θ – ℓn |sec θ + tan θ| + C

$= \sqrt{1+x^2}\;\; \tan^{-1}x - \log\left|\; x+\sqrt{1+x^2}\;\right| + C$

Hindi. $\int \frac{x\tan^{-1}x \quad dx}{\sqrt{1+x^2}}$ Put x = tan θ ⇒ dx = sec²θ dθ

⇒ I = $\int \frac{\tan\theta \;.\; \theta}{\sec\theta}\;.\sec^2\theta \quad d\theta \;= \theta \int \tan\theta\sec\theta \quad d\theta - \int 1 \;.\; \sec\theta \quad d\theta$ = θ sec θ – ℓn |sec θ + tan θ| + C

$= \sqrt{1+x^2}\;\; \tan^{-1}x - \log\left|\; x+\sqrt{1+x^2}\;\right| + C$

16. $\int \frac{x+\sqrt{x+1}}{x+2}$ dx is equal to **[16JM120494]**

(A) $(x+1) - 2\sqrt{x+1} + 2\,\ell n|x+2| - 2\tan^{-1}\sqrt{x+1} + C$

(B) $(x+1) + 2\sqrt{x+2} - 2\,\ell n|x+2| - 2\tan^{-1}\sqrt{x+2} + C$

(C*) $(x+1) + 2\sqrt{x+1} - 2\,\ell n|x+2| - 2\tan^{-1}\sqrt{x+1} + C$

(D) $(x+1) + 2\sqrt{x+2} - 2\,\ell n|x+1| + 2\tan^{-1}\sqrt{x+2} + C$

Sol. I = $\int \frac{x+\sqrt{x+1}}{x+2}$ dx put x + 1 = t² रखने पर ⇒ dx = 2t dt

⇒ I = $\int \frac{t^2-1+t}{t^2-1+2}.2t \quad dt \;=\; \int \frac{2t^3+2t^2-2t}{t^2+1} \quad dt$

$= \int (2t+2) \quad dt - 2\int \frac{2t+1}{t^2+1}dt \;=\; t^2 + 2t - 2\,\ell n\,(t^2+1) - 2\tan^{-1}t + C$

$= (x+1) + 2\sqrt{x+1} - 2\,\ell n|x+2| - 2\tan^{-1}\sqrt{x+1} + C$

17. The value of $\int \sqrt{\frac{1-\cos x}{\cos\alpha - \cos x}}$ dx, where 0 < α < x < π, is equal to

$\int \sqrt{\frac{1-\cos x}{\cos\alpha - \cos x}}$ dx, जहाँ 0 < α < x < π का मान है –

(A) $2\,\ell n\left(\cos\frac{\alpha}{2} - \cos\frac{x}{2}\right) + C$

(B) $\sqrt{2}\,\ell n\left(\cos\frac{\alpha}{2} - \cos\frac{x}{2}\right) + C$

(C) $2\sqrt{2}\,\ell n\left(\cos\frac{\alpha}{2} - \cos\frac{x}{2}\right) + C$

(D*) $-2\sin^{-1}\left(\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}}\right) + C$

Sol. I = $\int \sqrt{\frac{1-\cos x}{\cos\alpha - \cos x}}$ dx   0 < α < x < π

$= \int \frac{\sqrt{2}\sin\frac{x}{2}dx}{\sqrt{2\cos^2\frac{\alpha}{2} - 1 - 2\cos^2\frac{x}{2} + 1}} = \int \frac{\sin\frac{x}{2}dx}{\sqrt{\cos^2\frac{\alpha}{2} - \cos^2\frac{x}{2}}}$ put $\cos\frac{x}{2}$ = t रखने पर ⇒ $-\frac{1}{2}\sin\frac{x}{2}$ dx = dt

$\Rightarrow \quad I = \int \frac{-2\ dt}{\sqrt{\cos^2\frac{\alpha}{2} - t^2}} \quad = -2\sin^{-1}\left(\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}}\right) + C$

18. If $I = \int \frac{\sin x + \sin^3 x}{\cos 2x}\ dx = A\cos x + B\ \ell n\ |f(x)| + C$, then

यदि $I = \int \frac{\sin x + \sin^3 x}{\cos 2x}\ dx = A\cos x + B\ \ell n\ |f(x)| + C$ हो, तो–

(A) $A = \frac{1}{4}$, $B = \frac{-1}{\sqrt{2}}$, $f(x) = \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}$ (B) $A = -\frac{1}{2}$, $B = \frac{-3}{4\sqrt{2}}$, $f(x) = \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}$

(C) $A = -\frac{1}{2}$, $B = \frac{3}{\sqrt{2}}$, $f(x) = \frac{\sqrt{2}\cos x + 1}{\sqrt{2}\cos x - 1}$ (D*) $A = \frac{1}{2}$, $B = \frac{-3}{4\sqrt{2}}$, $f(x) = \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}$

Sol. $I = \int \frac{\sin x(1+\sin^2 x)}{2\cos^2 x - 1}$ put $\cos x = t$ रखने पर $= -\sin x\ dx = dt$

$I = -\int \frac{2-t^2}{2t^2-1}\ dt = \int \frac{t^2-2}{2t^2-1}\ dt = \frac{1}{2}\int \frac{2t^2-4}{2t^2-1}\ dt = \frac{1}{2}\int dt - \frac{3}{2}\int \frac{dt}{2t^2-1}$

$= \frac{t}{2} - \frac{3}{2\sqrt{2}} \cdot \frac{1}{2}\ \ell n \left| \frac{\sqrt{2}\ t-1}{\sqrt{2}\ t+1} \right| + C = \frac{1}{2}\cos x - \frac{3}{4\sqrt{2}}\ \ell n \left| \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1} \right| + C$

$A = \frac{1}{2}$, $B = \frac{-3}{4\sqrt{2}}$, $f(x) = \ell n \left| \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1} \right|$

or (या) $A = \frac{1}{2}$, $B = \frac{3}{4\sqrt{2}}$, $f(x) = \ell n \left| \frac{\sqrt{2}\cos x + 1}{\sqrt{2}\cos x - 1} \right|$

19. The value of $\int \frac{1}{\cos^6 x + \sin^6 x}\ dx$ is equal to

$\int \frac{1}{\cos^6 x + \sin^6 x}\ dx$ का मान है–

(A) $\tan^{-1}(\tan x + \cot x) + C$ (B) $-\tan^{-1}(\tan x + \cot x) + C$

(C*) $\tan^{-1}(\tan x - \cot x) + C$ (D) $-\tan^{-1}(\tan x - \cot x) + C$

Sol. $I = \int \frac{1}{\cos^6 x + \sin^6 x}\ dx = \int \frac{1}{(\cos^2 x + \sin^2 x)\ (\cos^4 x + \sin^4 x - \cos^2 x \sin^2 x)}\ dx$

$= \int \frac{dx}{1 - 3\sin^2 x \cos^2 x} = \int \frac{\sec^4 x}{\sec^4 x - 3\tan^2 x}dx = \int \frac{(1+\tan^2 x)\sec^2 x}{(1+\tan^2 x)^2 - 3\tan^2 x}dx$

$= \int \frac{\sec^2 x(1+\tan^2 x)}{\tan^4 x - \tan^2 x + 1}\ dx$

$= \int \frac{\left(1 + \frac{1}{\tan^2 x}\right)\ \sec^2 x}{\left(\tan^2 x + \frac{1}{\tan^2 x} - 2\right) + 1}\ dx$ Put $\tan x = t$ रखने पर $\Rightarrow$ $\sec^2 x\ dx = dt$

$\Rightarrow$ $I = \int \frac{\left(1+\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}-2\right)+1}\,dt = \tan^{-1}\left(t-\frac{1}{t}\right) + C = \tan^{-1}(\tan x - \cot x) + C.$

20. Consider the following statements : **16JM120495]**

निम्नलिखित कथनों पर विचार कीजिए :

S_1 : The antiderivative of every even function is an odd function.

S_1 : प्रत्येक सम फलन का प्रतिअवकलज एक विषम फलन होता है।

S_2 : Primitive of $\frac{3x^4-1}{(x^4+x+1)^2}$ w.r.t. x is $\frac{x}{x^4+x+1}$ + C.

S_2 : $\frac{3x^4-1}{(x^4+x+1)^2}$ का x के सापेक्ष समाकलन $\frac{x}{x^4+x+1}$ + C है।

S_3 : $\int \frac{1}{\sqrt{\sin^3 x \cos x}}\,dx = \frac{-2}{\sqrt{\tan x}} + C.$

S_3 : $\int \frac{1}{\sqrt{\sin^3 x \cos x}}\,dx$ का मान $\frac{-2}{\sqrt{\tan x}}$ + C है।

S_4 : The value of $\int \left(\sqrt{\frac{a+x}{a-x}} - \sqrt{\frac{a-x}{a+x}}\right) dx$ is equal to $-2\sqrt{a^2-x^2} + C$

S_4 : $\int \left(\sqrt{\frac{a+x}{a-x}} - \sqrt{\frac{a-x}{a+x}}\right) dx$ का मान $-2\sqrt{a^2-x^2}$ + C है।

State, in order, whether S_1, S_2, S_3, S_4 are true or false

S_1, S_2, S_3, S_4 के सत्य (T) या असत्य (F) होने का सही क्रम है –

(A*) FFTT (B) TTTT (C) FFFF (D) TFTF

Sol. **S_1 :** Obvious स्पष्टतया

S_2 : If primitive of $\frac{3x^4-1}{(x^4+x+1)^2}$ is $\frac{x}{x^4+x+1}$ + C, then derivative of $\frac{x}{x^4+x+1}$ + C must be $\frac{3x^4-1}{(x^4+x+1)^2}$

Now

Let $f(x) = \frac{x}{x^4+x+1} + C$

$\Rightarrow f'(x) = \frac{(x^4+x+1)-x(4x^3+1)}{(x^4+x+1)^2} = \frac{-3x^4+1}{(x^4+x+1)^2}$

Hence given statement is false.

Hindi. यदि $\frac{3x^4-1}{(x^4+x+1)^2}$ का समाकलन $\frac{x}{x^4+x+1}$ + C है, तो $\frac{x}{x^4+x+1}$ + C का अवकलन $\frac{3x^4-1}{(x^4+x+1)^2}$ होना चाहिए।

अब माना $f(x) = \frac{x}{x^4+x+1} + C$

$\Rightarrow f'(x) = \frac{(x^4+x+1)-x(4x^3+1)}{(x^4+x+1)^2} = \frac{-3x^4+1}{(x^4+x+1)^2}$

इस प्रकार दिया गया कथन असत्य है।

S_3 : Let माना $I = \int \frac{dx}{\sqrt{\sin^3 x.\cos x}}$ $= \int \sin^{-3/2} x \cos^{-1/2} x \quad dx$

$= \int \tan^{-3/2} x.\sec^2 x \quad dx$ put $\tan x = t$ रखने पर $\Rightarrow$ $\sec^2 x\, dx = dt$

$\Rightarrow$ $I = \int t^{-3/2} dt = \frac{-2}{\sqrt{\tan x}} + C$

S_4 : Sol. Let माना $I = \int \left(\sqrt{\frac{a+x}{a-x}} - \sqrt{\frac{a-x}{a+x}} \right) dx$

$= \int \frac{2x}{\sqrt{a^2 - x^2}}\, dx = -\, 2\sqrt{a^2 - x^2} + C$

21. If $I_n = \int (\sin x + \cos x)^n dx$, snd $I_n = \frac{1}{n}(\sin x + \cos x)^{n-1} (\sin x - \cos x) + \frac{2k}{n} I_{n-2}$ then k =

(A) (n+ 1) (B*) (n – 1) (C) (2n+ 1) (A) (2n– 1)

यदि $I_n = \int (\sin x + \cos x)^n dx$, और $I_n = \frac{1}{n}(\sin x + \cos x)^{n-1} (\sin x - \cos x) + \frac{2k}{n} I_{n-2}$ तब k =

(A) (n+ 1) (B*) (n – 1) (C) (2n+ 1) (A) (2n– 1)

Sol. $I_n = \int (\sin x + \cos x)^{n-1} (\sin x + \cos x) dx$

$\Rightarrow$ $I_n = (\sin x + \cos x)^{n-1} (-\cos x + \sin x) - (n-1) \int (\sin x + \cos x)^{n-2} (\cos x - \sin x)(-\cos x + \sin x) dx$

$\Rightarrow$ $I_n = (\sin x + \cos x)^{n-1} (\sin x - \cos x) + (n-1)(\sin x + \cos x)^{n-2} (2 - (\sin x + \cos x)^2) dx$

$\Rightarrow$ $I_n = (\sin x + \cos x)^{n-1} (\sin x - \cos x) + 2(n-1) I_{n-2} - (n-1) I_n$

$\Rightarrow$ $nI_n = (\sin x + \cos x)^{n-1} (\sin x - \cos x) + 2(n-1) I_{n-2}$

$\Rightarrow$ $I_n = \frac{1}{n}(\sin x + \cos x)^{n-1} (\sin x - \cos x) + \frac{2(n-1)}{n} I_{n-2}$

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

भाग - II : एकल एवं द्वि–पूर्णांक मान प्रकार (SINGLE AND DOUBLE VALUE INTEGER TYPE)

*** In each question C is arbitrary constant**

1. If $f(x) = \int \frac{2\sin x - \sin 2x}{x^3} dx$, where $x \neq 0$, then $\underset{x \to 0}{\text{Limit}}\ f'(x)$ has the value

यदि $f(x) = \int \frac{2\sin x - \sin 2x}{x^3} dx$, जहाँ $x \neq 0$ हो, तो $\underset{x \to 0}{\text{Limit}}\ f'(x)$ का मान है–

Ans. **01.00**

Sol. $f(x) = \int \frac{2\sin x - \sin 2x}{x^3} dx$

$$f'(x) = \frac{2\sin x - \sin 2x}{x^3} = \frac{2\sin x}{x} \cdot \frac{1-\cos x}{x^2} = 2\left(\frac{\sin x}{x}\right) \cdot \frac{2\sin^2\frac{x}{2}}{\frac{x^2}{4}\times 4} = \frac{2\times 2}{4}\left(\frac{\sin x}{x}\right) \cdot \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(\frac{\sin x}{x}\right) \cdot \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = 1$$

2. If $\int \sin^4 x \cos^4 x\, dx = \frac{1}{k}\left[ax - \sin 4x + \frac{1}{8} . \sin 8x\right] + C$ then value of $\frac{k}{a}$ is

यदि $\int \sin^4 x \cos^4 x\, dx = \frac{1}{k}\left[ax - \sin 4x + \frac{1}{8} . \sin 8x\right] + C$ तब $\frac{k}{a}$ का मान है : **[16JM120496]**

Ans. **42.66 or 42.67**

Sol. $\int \sin^4 x \quad \cos^4 x \quad dx = \frac{1}{16}\int \sin^4 2x \quad dx = \frac{1}{16}\int\left(\frac{1-\cos 4x}{2}\right)^2 dx = \frac{1}{64}\int\left(1+\cos^2 4x - 2\cos 4x\right) dx$

$= \frac{1}{64}\int dx - \frac{1}{32}\int \cos 4x \quad dx + \frac{1}{64}\int\left(\frac{1+\cos 8x}{2}\right) dx$

$= \frac{x}{64} - \frac{\sin 4x}{128} + \frac{1}{64} \times \frac{x}{2} + \frac{1}{128}\frac{\sin 8x}{8} + C$

$= \frac{3}{128} x - \frac{1}{128} \sin 4x + \frac{\sin 8x}{128 \times 8} + C$

$= \frac{1}{128}\left[3x - \sin 4x + \frac{\sin 8x}{8}\right] + C$

3. Let f(x) be the primitive of $\frac{3x+2}{\sqrt{x-9}}$ w.r.t. to x. If f(10) = 60 then value of f(11) is.

माना $\frac{3x+2}{\sqrt{x-9}}$ का x के सापेक्ष समाकलन f(x) है। यदि f(10) = 60 तब f(11) का मान होगा –

Ans. **87.68**

Sol. $f(x) = \int \frac{3x+2}{\sqrt{x-9}} dx$ put $x - 9 = t^2$ रखने पर

$= 2\int (29 + 3t^2)\, dt$

$f(x) = 2(29\sqrt{x-9} + (x-9)^{3/2}) + C$

$\therefore \quad f(10) = 60 \quad \Rightarrow \quad C = 0$

$f(x) = 2(29\sqrt{x-9} + (x-9)^{3/2})$

$f(11) = 62\sqrt{2}$

4. If $\int \frac{\sqrt{4+x^2}}{x^6}\,dx = \frac{\left(a+x^2\right)^{3/2}.\left(x^2-b\right)}{120\,x^5}$ + C then $a^2 + b^2$ equals to :

[16JM120497]

यदि $\int \frac{\sqrt{4+x^2}}{x^6}\,dx = \frac{\left(a+x^2\right)^{3/2}.\left(x^2-b\right)}{120\,x^5}$ + C तब $a^2 + b^2$ बराबर है

Ans. **52.00**

Sol. $I = \int \frac{\sqrt{4+x^2}}{x^6}\,dx$ Put $x = 2\tan\theta$ रखने पर $\Rightarrow$ $dx = 2\sec^2\theta\, d\theta$

$\Rightarrow \quad I = \int \frac{2\sec\theta.2\sec^2\theta\ \ d\theta}{2^6\tan^6\theta} = \frac{1}{2^4}\int \frac{\cos^3\theta\ \ d\theta}{\sin^6\theta}$

$= \frac{1}{2^4}\int \left(\frac{1-\sin^2\theta}{\sin^6\theta}\right)\cos\theta\, d\theta$ put $\sin\theta = t$ रखने पर $\Rightarrow$ $\cos\theta\, d\theta = dt$

$= \frac{1}{2^4}\int \frac{dt}{t^6} - \int \frac{1}{t^4}dt = \frac{1}{16}\left[\frac{1}{3t^3} - \frac{1}{5t^5}\right] + C = \frac{1}{16}\left(\frac{5\sin^2\theta - 3}{15\sin^5\theta}\right) + C$

but लेकिन $\tan\theta = \frac{x}{2}$ So इसलिए $\sin\theta = \frac{x}{\sqrt{4+x^2}}$

$= \frac{1}{16}\left(\frac{\frac{5x^2}{x^2+4} - 3}{\frac{15x^5}{\left(x^2+4\right)^{5/2}}}\right) + C = \frac{1}{16}\left(\frac{\left(2x^2-12\right)\ \left(x^2+4\right)^{3/2}}{15x^5}\right) + C = \frac{1}{120}\left(\frac{\left(x^2+4\right)^{3/2}\ \left(x^2-6\right)}{x^5}\right) + C$

5. If $\int \sqrt{\frac{x}{a^3 - x^3}}\,dx = k\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)$ + C, then k equals to.

यदि $\int \sqrt{\frac{x}{a^3 - x^3}}\,dx = k\ \sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)$ + C, तब k का मान होगा –

Ans. **00.66 or 00.67**

Sol. $I = \int \sqrt{\frac{x}{a^3 - x^3}}\ dx$ put $x^3 = a^3\sin^2\theta$ रखने पर $\Rightarrow$ $x = a\sin^{2/3}\theta \Rightarrow dx = a \times \frac{2}{3}\frac{\cos\theta}{\sin^{1/3}\theta}\,d\theta$

$\Rightarrow \quad I = \int \sqrt{\frac{a\sin^{2/3}\theta}{a^3\left(1-\sin^2\theta\right)}} \times \frac{2a}{3}\left(\frac{\cos\theta}{\sin^{1/3}\theta}\right)d\theta = \int \frac{\sin^{1/3}\theta}{a\cos\theta} \times \frac{2a}{3}\ \frac{\cos\theta}{\sin^{1/3}\theta}d\theta = \frac{2}{3}\ \theta + C$

$= \frac{2}{3}\sin^{-1}\left(\left(\frac{x}{a}\right)^{3/2}\right) + C$

6. If $\int \frac{x\ dx}{\sqrt{1 + x^2 + \sqrt{\left(1+x^2\right)^3}}} = k\sqrt{1 + \sqrt{1+x^2}}$ + C then k^4 equals to : **[16JM120498]**

यदि $\int \frac{x\ dx}{\sqrt{1 + x^2 + \sqrt{\left(1+x^2\right)^3}}} = k\sqrt{1 + \sqrt{1+x^2}}$ + C तब k^4 का मान होगा –

Ans. **16.00**

Sol. $I = \int \frac{x\ dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$ Put $1 + x^2 = t^2$, then $2x\,dx = 2t\,dt$

$\Rightarrow \quad I = \int \frac{t\ dt}{\sqrt{t^2+t^3}} = \int \frac{dt}{\sqrt{1+t}} = 2\sqrt{1+t} = 2\sqrt{1+\sqrt{1+x^2}} + C$

Hindi $I = \int \frac{x\ dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$ $1 + x^2 = t^2$ रखने पर $2x\,dx = 2t\,dt$

$\Rightarrow \quad I = \int \frac{t\ dt}{\sqrt{t^2+t^3}} = \int \frac{dt}{\sqrt{1+t}} = 2\sqrt{1+t} = 2\sqrt{1+\sqrt{1+x^2}} + C$

7. If $\int e^{\sin x}.\frac{x\cos^3 x - \sin x}{\cos^2 x}\,dx = e^{\sin x} f(x) + C$ such that $f(0) = -1$ then value of $f(4\pi)$ is

यदि $\int e^{\sin x}.\frac{x\cos^3 x - \sin x}{\cos^2 x}\,dx = e^{\sin x} f(x) + C$ इस प्रकार है कि $f(0) = -1$ तब $f(4\pi)$ का मान होगा –

Ans. **11.56 or 11.57**

Sol. $\int e^{\sin x}.\frac{x\cos^3 x - \sin x}{\cos^2 x}\,dx = \int e^{\sin x}(x\cos x - \tan x \sec x)\ dx =$

$\int e^{\sin x}.\ x\cos x\ dx - \int e^{\sin x}\tan x \sec x dx$

(by parts integration) (खण्डशः समाकलन से)

$= xe^{\sin x} - \int 1.\ e^{\sin x}dx - [e^{\sin x}\sec x - \int e^{\sin x}\cos x.\sec x dx] + C$

$= e^{\sin x}(x - \sec x) + C \quad \therefore \quad f(x) = x - \sec x$

$f(4\pi) = 4\pi - 1$

8. Let $g(x) = \int \frac{1+2\cos x}{(\cos x+2)^2}dx$ and $g(0) = 0$ then value of $g\left(\frac{\pi}{2}\right)$ is. **[16JM120499]**

माना $g(x) = \int \frac{1+2\cos x}{(\cos x+2)^2}dx$ तथा $g(0) = 0$ तब $g\left(\frac{\pi}{2}\right)$ बराबर है

Ans. **00.50**

Sol. $g(x) = \int \frac{1+2\cos x}{(\cos x+2)^2}dx$; $g(x) = \int \frac{\cos x(\cos x+2)+\sin^2 x}{(\cos x+2)^2}dx$

$g(x) = \int \cos x.\frac{1}{(\cos x+2)}dx + \int \frac{\sin^2 x}{(\cos x+2)^2}dx$

$= \frac{1}{(\cos x+2)}.\sin x - \int -\frac{(-\sin x)}{(\cos x+2)^2}.\sin x dx + \int \frac{\sin^2 x}{(\cos x+2)^2}dx$

$g(x) = \frac{\sin x}{(\cos x+2)} + C$

$g(0) = 0 \quad \Rightarrow \quad C = 0$

$g(x) = \frac{\sin x}{\cos x+2}$

$g\left(\frac{\pi}{2}\right) = \frac{1}{2}$

9. If $f(x) = \sqrt{x-1}$; $g(x) = e^x$ and $\int fog(x)dx = Afog(x) + B\tan^{-1}(fog(x)) + C$ then $A^3 + B^2$ equals

यदि $f(x) = \sqrt{x-1}$; $g(x) = e^x$ और $\int fog(x)dx = Afog(x) + B\tan^{-1}(fog(x)) + C$ तब $A^3 + B^2$ बराबर है

Ans. **12.00**

Sol. $fog(x) = \sqrt{e^x - 1}$

$\int fog(x)dx = \int \sqrt{e^x - 1}\, dx$ put $e^x - 1 = t^2$

$= \int \frac{2t^2}{1+t^2} dt = 2t - 2\tan^{-1}t + C = 2\sqrt{e^x-1} - 2\tan^{-1}\sqrt{e^x-1} + C$

$A^3 + B^2 = 2^3 + (-2)^2 = 12$

10. If $\int \frac{2\sin 2\phi - \cos\phi}{6 - \cos^2\phi - 4\sin\phi} d\phi = p\,\ell n\left|\sin^2\phi - 4\sin\phi + 5\right| + q\tan^{-1}(\sin\phi - r) + C$ then value of $\frac{p^2+q^2}{r}$ is

[16JM120500]

यदि $\int \frac{2\sin 2\phi - \cos\phi}{6 - \cos^2\phi - 4\sin\phi} d\phi = p\,\ell n\left|\sin^2\phi - 4\sin\phi + 5\right| + q\tan^{-1}(\sin\phi - r) + C$ तब $\frac{p^2+q^2}{r}$ का मान बराबर है –

Ans. **26.50**

Sol. $I = \int \frac{2\sin 2\phi - \cos\phi}{6 - \cos^2\phi - 4\sin\phi} d\phi$

$= \int \frac{(4\sin\phi - 1)\cos\phi}{6 - (1 - \sin^2\phi) - 4\sin\phi} d\phi$ put $\sin\phi = t$ रखने पर $\Rightarrow$ $\cos\phi\, d\phi = dt$

$\Rightarrow \quad I = \int \frac{(4t-1)\,dt}{5 + t^2 - 4t} = 2\int \frac{(2t-4) + 7/2}{t^2 - 4t + 5} dt = 2[\ell n|t^2 - 4t + 5|] + 7\int \frac{1}{(t-2)^2 + (1)^2} dt$

$= 2[\ell n|t^2 - 4t + 5|] + 7\tan^{-1}\left(\frac{t-2}{1}\right) + C = 2\,\ell n\left|\sin^2\phi - 4\sin\phi + 5\right| + 7\tan^{-1}(\sin\phi - 2) + C$

11. If $\int \frac{(x-1)^2}{x^4 + x^2 + 1} dx = \frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right) - \frac{b}{\sqrt{a}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C$ then $a^2 + b^2$ equals to :

यदि $\int \frac{(x-1)^2}{x^4 + x^2 + 1} dx = \frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right) - \frac{b}{\sqrt{a}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C$ तब $a^2 + b^2$ बराबर है

Ans. **13.00**

Sol. $I = \int \frac{(x-1)^2}{x^4+x^2+1} dx = \int \frac{x^2+1}{x^4+x^2+1} dx - \int \frac{2x}{x^4+x^2+1} dx = I_1 - I_2$

Now अब $I_1 = \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx$ put $x - \frac{1}{x} = t$ रखने पर $\Rightarrow$ $\left(1 + \frac{1}{x^2}\right) dx = dt$

$\Rightarrow \quad I_1 = \int \frac{dt}{t^2+3} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) + C_1$

Also तथा $I_2 = \int \frac{2x}{x^4+x^2+1} dx$ put $x^2 = t$ रखने पर $\Rightarrow$ $2x\,dx = dt$

$$\Rightarrow \quad I_2 = \int \frac{dt}{t^2+t+1} = \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C_2$$

$$\int \frac{(x-1)^2}{x^4+x^2+1} dx = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right) - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$$

12. If $\int \frac{1+x\cos x}{x\left(1-x^2 e^{2\sin x}\right)} dx = k\,\ell n \sqrt{\frac{x^2 e^{2\sin x}}{1-x^2 e^{2\sin x}}} + C$ then k is equal to : **[16JM120501]**

यदि $\int \frac{1+x\cos x}{x\left(1-x^2 e^{2\sin x}\right)} dx = k\,\ell n \sqrt{\frac{x^2 e^{2\sin x}}{1-x^2 e^{2\sin x}}} + C$ तब k बराबर है

Ans. **01.00**

Sol. $\int \frac{1+x\cos x}{x\left(1-x^2 e^{2\sin x}\right)} dx$

Put $xe^{\sin x} = t$ (रखने पर) $\Rightarrow (xe^{\sin x}.\cos x + e^{\sin x})\,dx = dt \Rightarrow e^{\sin x}(x\cos x + 1)\,dx = dt$

$$\Rightarrow \quad I = \int \frac{dt}{t(1-t^2)} = \int \frac{1}{t(1-t)(1+t)} dt$$

Let (माना) $\frac{1}{t(1-t)(1+t)} = \frac{A}{t} + \frac{B}{(1-t)} + \frac{C}{1+t}$

$1 = A(1-t)(1+t) + B(t)(1+t) + C(t)(1-t)$

put t = 0 (रखने पर) $\Rightarrow$ A = 1

Put t = 1 (रखने पर) $\Rightarrow$ B = 1/2

Put t = –1 (रखने पर) $\Rightarrow$ C = –1/2

$$\Rightarrow I = \int \left\{\frac{1}{t} + \frac{1}{2(1-t)} - \frac{1}{2(1+t)}\right\} dt = \ell n|t| - \frac{1}{2}\ell n|1-t| - \frac{1}{2}\ell n|1+t| + C$$

$$= \ell n|x e^{\sin x}| - \frac{1}{2}\log|1 - x^2 e^{2\sin x}| + C = \ell n \sqrt{\frac{x^2 e^{2\sin x}}{1-x^2 e^{2\sin x}}} + C$$

13. If $\int \frac{x^4+1}{x\left(x^2+1\right)^2} dx = A\,\ell n\,|x| + \frac{B}{1+x^2} + C$, then A + B equals to :

यदि $\int \frac{x^4+1}{x\left(x^2+1\right)^2} dx = A\,\ell n\,|x| + \frac{B}{1+x^2} + C$, तब A + B का मान है–

Ans. **02.00**

Sol. $\int \frac{x^4+1}{x(x^2+1)^2} dx = A\,\ell n\,|x| + \frac{B}{1+x^2} + C$

$$\int \frac{x^4+1}{x(x^2+1)^2} dx = \int \frac{(x^2+1)^2 - 2x^2}{x(x^2+1)^2} dx = \int \frac{1}{x} dx - \int \frac{2x}{(x^2+1)^2} dx = \ell n|x| + \frac{1}{1+x^2} + C$$

Hindi $\int \frac{x^4+1}{x(x^2+1)^2} dx = A\,\ell n\,|x| + \frac{B}{1+x^2} + C$

$$\int \frac{x^4+1}{x(x^2+1)^2} dx = \int \frac{(x^2+1)^2 - 2x^2}{x(x^2+1)^2} dx = \int \frac{1}{x} dx - \int \frac{2x}{(x^2+1)^2} dx = \ell n|x| + \frac{1}{1+x^2} + C$$

14. If $\int \frac{1}{1-\sin^4 x} dx = \frac{1}{a\sqrt{b}} \tan^{-1}\left(\sqrt{a}\tan x\right) + \frac{1}{b}\tan x + C$ then value of $a^3 + b^3$ is:
[16JM120502]

यदि $\int \frac{1}{1-\sin^4 x} dx = \frac{1}{a\sqrt{b}} \tan^{-1}\left(\sqrt{a}\tan x\right) + \frac{1}{b}\tan x + C$ तब $a^3 + b^3$ का मान होगा –

Ans. **16.00**

Sol. $I = \int \frac{1}{1-\sin^4 x} dx = \int \frac{\sec^4 x}{\sec^4 x - \tan^4 x} dx = \int \frac{\sec^2 x.\ \sec^2 x\ dx}{\sec^2 x + \tan^2 x}$

$= \int \frac{(1+\tan^2 x)\ \sec^2 x\ dx}{2\tan^2 x + 1}$ put $\tan x = t$ रखने पर $\Rightarrow$ $\sec^2 x\, dx = dt$

$\Rightarrow$ $I = \frac{1}{2}\int \frac{2+2t^2}{2t^2+1} dt = \frac{1}{2}\left[\int dt + \int \frac{1}{2t^2+1} dt\right] = \frac{1}{2}t + \frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\ t) + C$

$= \frac{1}{2}\tan x + \frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + C$

15. If $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx = p \sin x - \frac{q}{\sin x} - r \tan^{-1}(\sin x) + C$ then value of $\frac{p+2r}{q}$ is

यदि $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx = p \sin x - \frac{q}{\sin x} - r \tan^{-1}(\sin x) + C$ तब $\frac{p+2r}{q}$ का मान होगा –

Ans. **06.50**

Sol. $I = \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx = \int \frac{(2-3\sin^2 x + \sin^4 x)\cos x}{\sin^2 x(1+\sin^2 x)} dx$

Let माना $\sin x = t$

$I = \int \frac{2-3t^2+t^4}{t^2+t^4} = \int\left(1+\frac{2}{t^2} - \frac{6}{t^2+1}\right)dt = t - \frac{2}{t} - 6\tan^{-1}(t) + c = \sin x - \frac{2}{\sin x} - 6\tan^{-1}(\sin x) + C$

16. If $\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}} = a\sqrt{\cot x} + b\sqrt{\tan^3 x} + C$, where C is an arbitrary constant of integration, then the values of $a^2 + 10b$ equals to : **[16JM120503]**

यदि $\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}} = a\sqrt{\cot x} + b\sqrt{\tan^3 x} + C$, जहाँ C स्वेच्छ अचर है तब $a^2 + 10b$ का मान बराबर है

Ans. **10.66 or 1067**

Sol. $I = \int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}} = \int \frac{dx}{\sqrt{\tan^3 x \cos^8 x}} = \int \frac{\sec^4 x\ dx}{\sqrt{\tan^3 x}}$

$= \int \frac{(1+\tan^2 x)\ \sec^2 x}{\sqrt{\tan^3 x}} dx$ Let (माना) $\tan x = t^{2/3}$ $\Rightarrow$ $\sec^2 x\, dx = \frac{2}{3}t^{-1/3}dt$

$\Rightarrow$ $I = \int \frac{(1+t^{4/3})}{t}\frac{2}{3}t^{-1/3}dt = \frac{2}{3}\int (1+t^{4/3})\ .t^{-4/3}dt = \frac{2}{3}\int (t^{-4/3}+1)\ dt = \frac{2}{3}\left[\frac{t^{-1/3}}{(-1/3)} + t\right] + C$

$= -2\ t^{-1/3} + \frac{2}{3}t + C = -2\sqrt{\cot x} + \frac{2}{3}\tan^{3/2} x + C$ $\Rightarrow$ $a = -2,\ b = 2/3$

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

भाग - III:एक या एक से अधिक सही विकल्प प्रकार (ONE OR MORE THAN ONE OPTIONS CORRECT TYPE)

*** In each question C is arbitrary constant**
प्रत्येक प्रश्न में C स्वेच्छ अचर है।

1. The value of $\int 2^{mx}.\,3^{nx}\,dx$ (when m, n $\in$ N) is equal to :
$\int 2^{mx}.\,3^{nx}\,dx$ (जब m, n $\in$ N) का मान है–

(A) $\dfrac{2^{mx}+3^{nx}}{m\ell n2+n\ell n3}+C$ (B*) $\dfrac{e^{(m\ell n2+n\ell n3)\,x}}{m\ell n2+n\ell n3}+C$ (C*) $\dfrac{2^{mx}\,.\,3^{nx}}{\ell n\left(2^{m}.3^{n}\right)}+C$ (D) $\dfrac{(mn)\,.\,2^{x}\,.\,3^{x}}{m\ell n2+n\ell n3}+C$

Sol. $I = \int 2^{mx}.\ 3^{nx}dx = \int \left(2^{m}.\ 3^{n}\right)^{x}dx = \dfrac{2^{mx}.\ 3^{nx}}{\ell n\ (2^{m}.\ 3^{n})}+C$

2. If $f\left(\dfrac{1-x}{1+x}\right) = x$ and $g(x) = \int f(x)\,dx$ then **[16JM120504]**

(A*) g(x) is continuous in domain (B) g(x) is discontinuous at two points in its domain

(C*) $\lim\limits_{x\to\infty} g'(x) = -1$ (D) $\int g(x)dx = -\dfrac{x^2}{2}+(2x+1)\,\lambda n\left(\dfrac{1+x}{e}\right)+C$

यदि $f\left(\dfrac{1-x}{1+x}\right) = x$ और $g(x) = \int f(x)\ dx$ तब

(A*) g(x), प्रान्त में सतत् है। (B) g(x), इसके प्रान्त में दो बिन्दुओं पर असतत् है।

(C*) $\lim\limits_{x\to\infty} g'(x) = -1$ (D) $\int g(x)dx = -\dfrac{x^2}{2}+(2x+1)\,\lambda n\left(\dfrac{1+x}{e}\right)+C$

Sol. $f\left(\dfrac{1-x}{1+x}\right) = x \quad \Rightarrow \quad f(x) = \dfrac{1-x}{1+x}$

$g(x) = \int f(x)dx = \int \dfrac{1-x}{1+x}dx$

$g(x) = -x + 2\ell n|1+x| + c$; $\int g(x)dx = -\dfrac{x^2}{2}+2x\ell n(x+1)-2x+2\ell n(x+1)+c$

$g'(x) = \dfrac{1-x}{1+x}$

$\lim\limits_{x\to\infty} g'(x) = -1$

3. If $= \int \tan^5 x dx = A\tan^4 x - \dfrac{1}{2}\tan^2 x + B\ell n|\sec x| + C$ then

यदि $= \int \tan^5 x dx = A\tan^4 x - \dfrac{1}{2}\tan^2 x + B\ell n|\sec x| + C$ तब

(A*) $A = \dfrac{1}{4}$ (B) $A = \dfrac{1}{2}$ (C*) B = 1 (D) B = – 1

Sol. $\int \tan^5 x dx = \int \tan^3 x\left(\sec^2 x-1\right)dx$

$= \int \tan^3 x\sec^2 x dx - \int \tan x\left(\sec^2 x-1\right)dx = \dfrac{\tan^4 x}{4} - \dfrac{\tan^2 x}{2} + \ell n|\sec x| + C$

4. The value of $\int \{1+2\tan x(\tan x+\sec x)\}^{1/2}\,dx$ is equal to **[16JM120505]**

$\int \{1+2\tan x(\tan x+\sec x)\}^{1/2}\ dx$ का मान है–

(A) $\ell n\ |\sec x\ (\sec x - \tan x)| + C$ (B) $\ell n\ |\text{cosec}\ x\ (\sec x + \tan x)| + C$

(C*) $\ell n\ |\sec x\ (\sec x + \tan x)| + C$ (D*) $-\ell n\ |\cos x(\sec x - \tan x)| + C$

Sol. $\int \{\sec^2 x+\tan^2 x+2\tan x\sec x\}^{1/2}\ dx$

$= \int (\sec x+\tan x)\ dx$

$= \ell n\ |(\sec x + \tan x)| + \ell n\ |\sec x| + C$

$= \ell n\ |\sec x\ (\sec x + \tan x)| + C$

5. The value of $\int \dfrac{\ell n\left(\dfrac{x-1}{x+1}\right)}{x^2-1}\ dx$ is equal to

$\int \dfrac{\ell n\left(\dfrac{x-1}{x+1}\right)}{x^2-1}\ dx$ का मान है–

(A) $\dfrac{1}{2}\ \ell n^2\ \dfrac{x-1}{x+1} + C$ (B*) $\dfrac{1}{4}\ \ell n^2\ \dfrac{x-1}{x+1} + C$ (C) $\dfrac{1}{2}\ \ell n^2\ \dfrac{x+1}{x-1} + C$ (D*) $\dfrac{1}{4}\ \ell n^2\ \dfrac{x+1}{x-1} + C$

Sol. $I = \int \dfrac{\ell n\left(\dfrac{x-1}{x+1}\right)}{x^2-1}\ dx$ put $\ell n\left(\dfrac{x-1}{x+1}\right) = t$ रखने पर $\Rightarrow \dfrac{2}{x^2-1}\ dx = dt$

$\Rightarrow \quad I = \int t\dfrac{dt}{2} = \dfrac{t^2}{4} + C$

$= \log^2\left(\dfrac{x-1}{x+1}\right) + C = \dfrac{1}{4}\ \log^2\left(\dfrac{x+1}{x-1}\right) + C$

6. The value of $\int \dfrac{\ell n\ (\tan x)}{\sin x\ \cos x}\ dx$ is equal to

$\int \dfrac{\ell n\ (\tan x)}{\sin x\ \cos x}\ dx$ का मान है–

(A*) $\dfrac{1}{2}\ \ell n^2\ (\cot x) + C$ (B) $\dfrac{1}{2}\ \ell n^2\ (\sec x) + C$

(C*) $\dfrac{1}{2}\ \ell n^2\ (\sin x\ \sec x) + C$ (D*) $\dfrac{1}{2}\ \ell n^2\ (\cos x\ \text{cosec}\ x) + C$

Sol. $I = \int \dfrac{\ell n\ (\tan x)}{\sin x\ \cos x}\ dx$ put $\ell n \tan x = t$ रखने पर $\Rightarrow \dfrac{1}{\sin x \cos x}dx = dt$

$I = \int t\ \ dt = \dfrac{t^2}{2} + C = \dfrac{1}{2}(\ell n \tan x)^2 + C = \dfrac{1}{2}(\ell n \cot x)^2 + C = \dfrac{1}{2}(\ell n^2 \cot x) + C$

$= \dfrac{1}{2}\ell n^2(\sin x \sec x) + C = \dfrac{1}{2}\ \ell n^2\ (\cos x\ \text{cosec}\ x) + C$

7. The value of $\int \dfrac{\cos^3 x}{\sin^2 x + \sin x}\ dx$ is equal to :

$\int \dfrac{\cos^3 x}{\sin^2 x + \sin x}\ dx$ का मान है–

(A) $\ell n\ |\sin x| + \sin x + C$ (B*) $\ell n\ |\sin x| - \sin x + C$

(C*) $-\ell n\ |\text{cosec } x| - \sin x + C$ (D) $-\ell n\ |\sin x| + \sin x + C$

Sol. $I = \int \frac{\cos^3 x}{\sin^2 x + \sin x} dx = \int \frac{\cos x\ .\ (1-\sin^2 x)}{\sin x(1+\sin x)} dx$ Put sin x = t , then cosx dx = dt

$\Rightarrow \quad I = \int \frac{(1-t)(1+t)\ \ dt}{t\ \ (1+t)} = \ell n\ |t| - t + C = \ell n\ |\sin x| - \sin x + C$

Hindi. $I = \int \frac{\cos^3 x}{\sin^2 x + \sin x} dx = \int \frac{\cos x\ .\ (1-\sin^2 x)}{\sin x(1+\sin x)} dx$

sin x = t रखने पर, cosx dx = dt

$= \int \frac{(1-t)(1+t)\ \ dt}{t\ \ (1+t)} = \ell n\ |t| - t + C = \ell n\ |\sin x| - \sin x + C$

8. If $\int \frac{(x-1)\, dx}{x^2\sqrt{2x^2-2x+1}} = \frac{\sqrt{f(x)}}{g(x)} + C$, where f(x) is a quadratic expression and g(x) is a monic linear expression. **[16JM120506]**

यदि $\int \frac{(x-1)\, dx}{x^2\sqrt{2x^2-2x+1}} = \frac{\sqrt{f(x)}}{g(x)} + C$ जहाँ f(x) द्विघात व्यंजक है तथा g(x) एक घातीय रैखिक व्यंजक है तब

(A*) $f(x) = 2x^2 - 2x + 1$ (B) $g(x) = x + 1$

(C*) $g(x) = x$ (D) $f(x) = 2x^2 - 2x$

Sol. $I = \int \frac{(x-1)\ \ dx}{x^2\sqrt{2x^2-2x+1}} = \int \frac{(x-1)\ \ dx}{x^3\sqrt{2-\frac{2}{x}+\frac{1}{x^2}}}$

Put $2 - \frac{2}{x} + \frac{1}{x^2} = t^2$, then $\left(\frac{x-1}{x^3}\right) dx = t\, dt$

$\Rightarrow \quad I = \int \frac{t\ \ dt}{t} = t + C = \sqrt{2-\frac{2}{x}+\frac{1}{x^2}} + C = \frac{\sqrt{2x^2-2x+1}}{x} + C$

So $\quad f(x) = \sqrt{2x^2-2x+1}$ and $g(x) = x$

Hindi $I = \int \frac{(x-1)\ \ dx}{x^2\sqrt{2x^2-2x+1}} = \int \frac{(x-1)\ \ dx}{x^3\sqrt{2-\frac{2}{x}+\frac{1}{x^2}}}$

माना $2 - \frac{2}{x} + \frac{1}{x^2} = t^2$, तो $\left(\frac{x-1}{x^3}\right) dx = t\, dt$

$\Rightarrow \quad I = \int \frac{t\ \ dt}{t} = t + C = \sqrt{2-\frac{2}{x}+\frac{1}{x^2}} + C = \frac{\sqrt{2x^2-2x+1}}{x} + C$

अतः $\quad f(x) = \sqrt{2x^2-2x+1}$ और $g(x) = x$

9. If $\int e^{3x} \cos 4x\, dx = e^{3x} (A \sin 4x + B \cos 4x) + C$ then :

यदि $\int e^{3x} \cos 4x\, dx = e^{3x} (A \sin 4x + B \cos 4x) + C$ हो, तो –

(A) 4A = 3B (B) 2A = 3B (C*) 3A = 4B (D*) 4A + 3B = 1

Sol. $I = \int e^{3x} \cos 4x\ \ dx = e^{3x}\left(A \sin 4x + B \cos 4x\right) + C$ ....(i)

$$I = \frac{1}{4}e^{3x}\ \sin 4x - \int \frac{3}{4}e^{3x}\sin 4x\ dx = \frac{1}{4}e^{3x}\sin 4x\ + \frac{3}{16}e^{3x}\cos 4x\ - \int \frac{9}{16}e^{3x}\cos 4x\, dx$$

$$\frac{25}{16}\ I = \frac{1}{16}\ \left(4e^{3x}\sin 4x + 3e^{3x}\cos 4x\right)$$

comparing with equation (i)
समीकरण (i) से तुलना करने पर

$\Rightarrow$ $A = \frac{4}{25}$, $B = \frac{3}{25}$ $\Rightarrow$ $\frac{A}{B} = \frac{4}{3} \Rightarrow$ $3A = 4B \Rightarrow$ $4A + 3B = 1$

10. $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$ dx equals to **[16JM120507]**

$I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$ dx बराबर है

(A*) $-x + \frac{2}{\pi}\ (2x - 1)\sin^{-1}\ \sqrt{x}\ + \ \frac{2}{\pi}\ \sqrt{x - x^2}\ + C$

(B*) $x - \ \frac{4x}{\pi}\ \cos^{-1}\ \sqrt{x} - \frac{2}{\pi}\sin^{-1}\ \sqrt{x}\ + \ \frac{2}{\pi}\ \sqrt{x}\ \sqrt{1-x}\ \ + C$

(C) $-x + \frac{2}{\pi}\ (2x + 1)\cos^{-1}\sqrt{x}\ + \frac{2}{\pi}\ \sqrt{x}\ \sqrt{1-x}\ \ + C$

(D) $x - \frac{4x}{\pi}\sin^{-1}\ \sqrt{x}\ + C$

Sol. $\because$ $\sin^{-1}\sqrt{x} + \cos^{-1}\ \sqrt{x}\ = \ \frac{\pi}{2}$

$\therefore$ $I = \int \frac{2}{\pi}\left(\frac{\pi}{2} - 2\cos^{-1}\sqrt{x}\right)dx$

$= x - \ \frac{4}{\pi}\ \int \cos^{-1}\sqrt{x}\ dx = x - \ \frac{4}{\pi}\left[\cos^{-1}\sqrt{x}.x - \int x.\frac{-1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}dx\right.$

$= x - \frac{4x}{\pi}\ \cos^{-1}\ \sqrt{x}\ - \ \frac{2}{\pi}\ \int \frac{\sqrt{x}}{\sqrt{1-x}}\ dx$

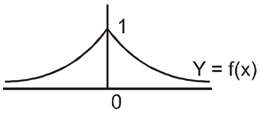
$= x - \ \frac{4x}{\pi}\ \cos^{-1}\ \sqrt{x}\ - \frac{2}{\pi}\ (\sin^{-1}\sqrt{x}\ - \sqrt{x}\ \sqrt{1-x}\ ) + C$

11. If $\int \frac{x^2 - x + 1}{\left(1 + x^2\right)^{3/2}}\ e^x dx = e^x f(x) + C$ then
(A*) f(x) is a an even function
(B*) f(x) is a bounded function
(C*) Range of f(x) is (0, 1]
(D) f(x) has two points of exterma.

यदि $\int \frac{x^2 - x + 1}{\left(1 + x^2\right)^{3/2}}\ e^x dx = e^x f(x) + C$ तब

(A*) f(x) समफलन है।
(B*) f(x) परिबद्ध फलन है।
(C*) f(x) का परिसर (0, 1] है।
(D) f(x) के चरम मान के दो बिन्दु है।

Sol. $I = \int e^x\left(\frac{1}{\sqrt{1+x^2}} - \frac{x}{\left(1+x^2\right)^{3/2}}\right)\ dx$



$I = \int e^x(f(x)+f'(x))dx = f(x)e^x + C$

$I = e^x.\ \dfrac{1}{\sqrt{1+x^2}} + C$

$f(x) = \dfrac{1}{\sqrt{1+x^2}}$

12. If $\int \dfrac{4e^x+6e^{-x}}{9e^x-4e^{-x}}\,dx = Ax + B\,\ell n\,|9e^{2x}-4| + C$, then **[16JM120508]**

यदि $\int \dfrac{4e^x+6e^{-x}}{9e^x-4e^{-x}}\,dx = Ax + B\,\ell n\,|9e^{2x}-4| + C$ हो, तो–

(A*) $A + 18B = 16$ (B*) $18B - A = 19$
(C) $A - 18B = 17$ (D) $A + 18B = 32$

Sol. $\int \dfrac{4e^x+6e^{-x}}{9e^x-4e^{-x}}.\ dx = Ax + B\ell n\,|9e^{2x}-4| + C$

put $4e^x + 6e^{-x} = P\left(9e^x-4e^{-x}\right) + Q\,(9e^x + 4e^{-x})$ रखने पर

$\Rightarrow\quad 4 = 9P + 9Q$ and तथा $6 = 4Q - 4P$

comparing, तुलना करने पर $P = -\dfrac{19}{36}$, $Q = \dfrac{35}{36}$

$I = -\dfrac{19}{36}\int dx + \dfrac{35}{36}\int \dfrac{9e^x+4e^{-x}}{9e^x-4e^{-x}}\,dx = -\dfrac{19}{36}.x + \dfrac{35}{36}\,\ell n\left|\left(9e^x-4e^{-x}\right)\right| + C$

$= -\dfrac{19}{36}x + \dfrac{35}{36}\,\ell n\left|\left(9e^{2x}-4\right)\right| - \dfrac{35}{36}x + C = \dfrac{35}{36}\,\ell n\left|\left(9e^{2x}-4\right)\right| - \dfrac{54}{36}x + C$

$= \dfrac{35}{36}\,\ell n\left|\left(9e^{2x}-4\right)\right| - \dfrac{3}{2}x + C$

So इसलिए $A = -\dfrac{3}{2}$, $B = \dfrac{35}{36}$, $C \in R$

13. The value of $\int \dfrac{x^2+\cos^2 x}{1+x^2}\,\text{cosec}^2 x\,dx$ is equal to:

$\int \dfrac{x^2+\cos^2 x}{1+x^2}\,\text{cosec}^2 x\,dx$ का मान है–

(A) $\cot x - \cot^{-1} x + C$ (B*) $C - \cot x + \cot^{-1} x$

(C*) $-\tan^{-1} x - \dfrac{\text{cosec}\,x}{\sec x} + C$ (D) $\dfrac{1}{\tan^{-1} x} - \cot x + C$

Sol. $\int \dfrac{x^2+\cos^2 x}{1+x^2}\,\text{cosec}^2 x\,dx = \int \text{cosec}^2 x dx - \int \dfrac{1}{1+x^2}dx = -\cot x - \tan^{-1}x + C$

$= -\cot x - \cot^{-1}x + C$ .... (1)

$= -\tan^{-1}x - \dfrac{\text{cosec}\,x}{\sec x} + C$ .... (2)

$= -e^{\ell n \tan^{-1}x} - \cot x + C$ .... (3)

14. The value of $\int \frac{dx}{\sqrt{x-x^2}}$; $\left(x > \frac{1}{2}\right)$ is equal to

$\int \frac{dx}{\sqrt{x-x^2}}$; $\left(x > \frac{1}{2}\right)$ का मान है–

(A*) $2\sin^{-1}\sqrt{x} + C$ (B*) $\sin^{-1}(2x-1) + C$

(C) $C - 2\cos^{-1}(2x-1)$ (D*) $\cos^{-1} 2\sqrt{x-x^2} + C$

Sol. $I = \int \frac{dx}{\sqrt{x-x^2}} = \int \frac{dx}{\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}}$

$= \sin^{-1}\frac{\left(x-\frac{1}{2}\right)}{1/2} = \sin^{-1}(2x-1) + C$ ..... (1)

Also तथा $I = \int \frac{dx}{\sqrt{x}\,\sqrt{1-x}}$ put $\sqrt{x} = t$ रखने पर $\Rightarrow$ $\frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow$ $I = \int \frac{2\,dt}{\sqrt{1-t^2}} = 2\sin^{-1}\sqrt{x} + C$ ....(2)

Now Let अब माना $\theta = \sin^{-1}(2x-1)$

$\Rightarrow$ $\sin\theta = 2x-1$ $\Rightarrow$ $\cos\theta = \sqrt{1-(2x-1)^2} = \sqrt{1-4x^2+4x-1} = 2\sqrt{x-x^2}$

$\Rightarrow$ $\theta = \cos^{-1} 2\sqrt{x-x^2}$

15. $\int \frac{x^3-1}{x^3+x}dx$ is equal to

(A) $x - \ell n|x| + \ell n(x^2+1) - \tan^{-1}x + C$

(B*) $x - \ell n|x| + \frac{1}{2}\ell n(x^2+1) - \tan^{-1}x + C$

(C) $x + \ell n|x| + \frac{1}{2}\ell n(x^2+1) + \tan^{-1}x + C$

(D*) $x + \ell n\sqrt{\frac{x^2+1}{x^2}} + \cot^{-1}x + C$

$\int \frac{x^3-1}{x^3+x}dx$ का मान है–

(A) $x - \ell n|x| + \ell n(x^2+1) - \tan^{-1}x + C$

(B*) $x - \ell n|x| + \frac{1}{2}\ell n(x^2+1) - \tan^{-1}x + C$

(C) $x + \ell n|x| + \frac{1}{2}\ell n(x^2+1) + \tan^{-1}x + C$

(D*) $x + \ell n\sqrt{\frac{x^2+1}{x^2}} + \cot^{-1}x + C$

Sol. $\int \frac{x^3-1}{x^3+x}dx = \int\left(1-\frac{1}{1+x^2}-\frac{1}{x}+\frac{x}{x^2+1}\right)dx$

$= x - \tan^{-1}x - \ell n|x| + \frac{1}{2}\ell n(x^2+1) + C$

16. The value of $2\int \sin x \,.\, \operatorname{cosec} 4x \; dx$ is equal to **[16JM120509]**

(A*) $\frac{1}{2\sqrt{2}}\ell n\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| - \frac{1}{4}\ell n\left|\frac{1+\sin x}{1-\sin x}\right| + C$

(B*) $\frac{1}{2\sqrt{2}}\ell n\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| - \frac{1}{2}\ell n\left|\frac{1+\sin x}{\cos x}\right| + C$

(C) $\frac{1}{2\sqrt{2}}\ell n\left|\frac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}\right| - \frac{1}{4}\ell n\left|\frac{1+\sin x}{1-\sin x}\right| + C$

(D*) $-\frac{1}{2\sqrt{2}}\ell n\left|\frac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}\right| + \frac{1}{4}\ell n\left|\frac{1-\sin x}{1+\sin x}\right| + C$

$2\int \sin x \,.\, \operatorname{cosec} 4x \; dx$ का मान है–

(A*) $\frac{1}{2\sqrt{2}}\ell n\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| - \frac{1}{4}\ell n\left|\frac{1+\sin x}{1-\sin x}\right| + C$

(B*) $\frac{1}{2\sqrt{2}}\ell n\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| - \frac{1}{2}\ell n\left|\frac{1+\sin x}{\cos x}\right| + C$

(C) $\frac{1}{2\sqrt{2}}\ell n\left|\frac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}\right| - \frac{1}{4}\ell n\left|\frac{1+\sin x}{1-\sin x}\right| + C$

(D*) $-\frac{1}{2\sqrt{2}}\ell n\left|\frac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}\right| + \frac{1}{4}\ell n\left|\frac{1-\sin x}{1+\sin x}\right| + C$

Sol. $I = 2\int \sin x \,.\, \operatorname{cosec} 4x \; dx = 2\int \frac{\sin x \; dx}{4\sin x \cos x \cos 2x}$

$= \frac{1}{2}\int \frac{\cos x dx}{(1-\sin^2 x)(1-2\sin^2 x)}$ Put sinx = t, then cos x dx = dt

$\Rightarrow I = \frac{1}{2}\int \frac{dt}{(1-t^2)(1-2t^2)}$

$= \frac{1}{2}\left[-\int \frac{1}{1-t^2}\,dt + \int \frac{2}{1-2t^2}\,dt\right]$ [By partial fraction]

$= \frac{1}{2}\left[-\frac{1}{2}\ell n\left|\frac{1+t}{1-t}\right| + \frac{2}{2\sqrt{2}}\ell n\left|\left(\frac{1+\sqrt{2}t}{1-\sqrt{2}\;t}\right)\right|\right] = \frac{1}{2\sqrt{2}}\ell n\left|\left(\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\;\sin x}\right)\right| - \frac{1}{4}\ell n\left|\left(\frac{1+\sin x}{1-\;\sin x}\right)\right| + C$

Hindi $I = 2\int \sin x \,.\, \operatorname{cosec} 4x \; dx = 2\int \frac{\sin x \; dx}{4\sin x \cos x \cos 2x}$

माना, sinx = t

तो cos x dx = dt

$= \frac{1}{2}\int \frac{\cos x dx}{(1-\sin^2 x)(1-2\sin^2 x)}$ [आंशिक भिन्न द्वारा]

$\Rightarrow I = \frac{1}{2}\int \frac{dt}{(1-t^2)(1-2t^2)}$

$= \frac{1}{2}\left[-\int\frac{1}{1-t^2}\ dt + \int\frac{2}{1-2t^2}\ dt\right]$ $= \frac{1}{2}\left[-\frac{1}{2}\ \ell n\left|\frac{1+t}{1-t}\right| + \frac{2}{2\sqrt{2}}\ell n\left|\left(\frac{1+\sqrt{2}t}{1-\sqrt{2}\ \ t}\right)\right|\right] = \frac{1}{2\sqrt{2}}\ \ell n$

$\left|\left(\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\ \ \sin x}\right)\right| - \frac{1}{4}\ell n\left|\left(\frac{1+\sin x}{1-\ \ \sin x}\right)\right| + C$

17. If $\int \frac{3\cot 3x - \cot x}{\tan x - 3\tan 3x}\ dx = p\ f(x) + q\ g(x) + C$, then which of the following may be correct?

यदि $\int \frac{3\cot 3x - \cot x}{\tan x - 3\tan 3x}\ dx = p\ f(x) + q\ g(x) + C$ तब–

(A*) $p = 1; q = \frac{1}{\sqrt{3}}$; $f(x) = x; g(x) = \ell n\left|\frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x}\right|$

(B) $p = 1; q = -\frac{1}{\sqrt{3}}$; $f(x) = x; g(x) = \ell n\left|\frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x}\right|$

(C) $p = 1; q = -\frac{2}{\sqrt{3}}$; $f(x) = x; g(x) = \ell n\left|\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right|$

(D*) $p = 1; q = -\frac{1}{\sqrt{3}}$; $f(x) = x; g(x) = \ell n\left|\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right|$

Sol. $I = \int\frac{3\cot 3x - \cot x}{\tan x - 3\tan 3x}\ dx = pf(x) + g(x) + C$ ....(i)

using $\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$ and $\cot 3x = \frac{\cot^3 x - 3\cot x}{3\cot^2 x - 1}$

$I = \int\frac{1 - 3\tan^2 x + 2 - 2}{3 - \tan^2 x}\ dx$

$I = \int 1\ .\ dx - 2\int\frac{1+\tan^2 x}{3 - \tan^2 x}\ dx = x - 2\int\frac{\sec^2 x}{3 - \tan^2 x}\ dx + C$

$= x - \frac{2}{2\sqrt{3}}\ \ell n\left|\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right| + C$

$= 1.x + \frac{1}{\sqrt{3}}\ell n\left|\frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x}\right| + C$

comparing with R.H.S. of equation (i), we get the result.

Hindi. $I = \int\frac{3\cot 3x - \cot x}{\tan x - 3\tan 3x}\ dx = pf(x) + g(x) + C$ ....(i)

$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$ और $\cot 3x = \frac{\cot^3 x - 3\cot x}{3\cot^2 x - 1}$ का प्रयोग करने पर

$I = \int\frac{1 - 3\tan^2 x + 2 - 2}{3 - \tan^2 x}\ dx$

$I = \int 1\ .\ dx - 2\int\frac{1+\tan^2 x}{3 - \tan^2 x}\ dx = x - 2\int\frac{\sec^2 x}{3 - \tan^2 x}\ dx + C$

$= x - \frac{2}{2\sqrt{3}}\ \ell n\left|\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right| + C$

$= 1.x + \frac{1}{\sqrt{3}} \ell n \left| \frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x} \right| + C$

तुलना करने पर p = 1, f(x) = x, q = $\frac{1}{\sqrt{3}}$ और

$g(x) = \ell n \left| \frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x} \right|$

18. If $\int \frac{dx}{5 + 4\cos x} = P \tan^{-1} \left(m \tan\frac{x}{2} \right) + C$ then :

यदि $\int \frac{dx}{5 + 4\cos x} = P \tan^{-1} \left(m \tan\frac{x}{2} \right) + C$ हो, तो –

(A*) P = 2/3 (B*) m = 1/3 (C) P = 1/3 (D) m = 2/3

Sol. Let माना $I = \int \frac{dx}{5+4\cos x} = I \tan^{-1} \left(\tan\frac{x}{2} \right) + C$

$\int \frac{dx}{1+4(1+\cos x)} = \int \frac{\sec^2\frac{x}{2}dx}{5+5\tan^2\frac{x}{2}+4-4\tan^2\frac{x}{2}}$

$= \int \frac{\sec^2\frac{x}{2}dx}{9+\tan^2\frac{x}{2}}$ Put $\tan\frac{x}{2} = t$ रखने पर $\Rightarrow \sec^2\frac{x}{2}\, dx = 2dt$

$\Rightarrow \quad I = 2\int \frac{dt}{9+t^2} = \frac{2}{3} \tan^{-1} \left(\frac{1}{3}\tan\frac{x}{2} \right) + C \Rightarrow \quad P = \frac{2}{3}, m = \frac{1}{3}$

19. The value of $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$ is equal to: **[16JM120510]**

$\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}$ dx का मान है–

(A*) $\cot^{-1}(\cot^2 x) + C$ (B*) $-\cot^{-1}(\tan^2 x) + C$
(C*) $\tan^{-1}(\tan^2 x) + C$ (D*) $-\tan^{-1}(\cos 2x) + C$

Sol. $I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx = \int \frac{2\tan x \sec^2 x}{\tan^4 x + 1} dx$ put $\tan^2 x = t$ रखने पर

$\Rightarrow 2 \tan x \sec^2 x\, dx = dt$

$= \int \frac{dt}{t^2+1} = \tan^{-1}(t) + C$

$= \tan^{-1}(\tan^2 x) + C \quad = \frac{\pi}{2} - \cot^{-1}\left(\tan^2 x\right) + C$

$= -\cot^{-1}(\tan^2 x) + C_1 \quad = -\cot^{-1}\left(\frac{1}{\cot^2 x}\right) + C_1$

$= \cot^{-1}(\cot^2 x) + C_1$

also तथा $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x} \Rightarrow \left(\frac{1-\cos 2x}{1+\cos 2x}\right) = \tan^2 x$, using these values in given integral

इन मानों का उपयोग दिये गये समाकल में करने पर

$$I = \int \frac{\sin 2x}{(\cos^2 x - \sin^2 x)^2 + 2\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin 2x}{(\cos 2x)^2 + \left(\frac{1-\cos^2 2x}{2}\right)} dx$$

$$= \int \frac{2\sin 2x}{(\cos 2x)^2 + 1} dx$$ put $\cos 2x = t$ रखने पर $\Rightarrow -2\sin 2x\, dx = dt$

$\Rightarrow \quad I = \int \frac{-dt}{t^2+1} = -\tan^{-1} t + C_2 = -\tan^{-1}(\cos 2x) + C_2$

PART - IV : COMPREHENSION

भाग - IV : अनुच्छेद (COMPREHENSION)

Comprehension # 1 (Q.No. 1 to 3)
अनुच्छेद # 1

Let $I_{n, m} = \int \sin^n x \cos^m x.dx$. Then we can relate $I_{n, m}$ with each of the following

(i)	$I_{n-2, m}$	(ii)	$I_{n+2, m}$	(iii)	$I_{n, m-2}$
(iv)	$I_{n, m+2}$	(v)	$I_{n-2, m+2}$	(vi)	$I_{n+2, m-2}$

Suppose we want to establish a relation between $I_{n, m}$ and $I_{n, m-2}$, then we set

$P(x) = \sin^{n+1}x \cos^{m-1}x$ ....(1)

In $I_{n,m}$ and $I_{n, m-2}$ the exponent of cosx is m and m – 2 respectively, the minimum of the two is m – 2, adding 1 to the minimum we get m – 2 + 1 = m – 1. Now choose the exponent m – 1 of cosx in P(x). Similarly choose the exponent of sin x for P (x)

Now differentiating both sides of (1), we get

$P'(x) = (n+1)\sin^n x \cos^m x - (m-1)\sin^{n+2}x \cos^{m-2}x$
$= (n+1)\sin^n x \cos^m x - (m-1)\sin^n x (1-\cos^2 x)\cos^{m-2}x$
$= (n+1)\sin^n x \cos^m x - (m-1)\sin^n x \cos^{m-2} x + (m-1)\sin^n x \cos^m x$
$= (n+m)\sin^n x \cos^m x - (m-1)\sin^n x \cos^{m-2}x$

Now integrating both sides, we get

$\sin^{n+1}x \cos^{m-1}x = (n+m) I_{n, m} - (m-1) I_{n, m-2}$.

Similarly we can establish the other relations.

अनुच्छेद # 1 (प्रश्न संख्या 1 से 3)

मानाकि $I_{n, m} = \int \sin^n x \cos^m x.dx$ है, तो हम $I_{n, m}$ को निम्न में से प्रत्येक के साथ सम्बन्धित कर सकते है–

(i)	$I_{n-2, m}$	(ii)	$I_{n+2, m}$	(iii)	$I_{n, m-2}$
(iv)	$I_{n, m+2}$	(v)	$I_{n-2, m+2}$	(vi)	$I_{n+2, m-2}$

मानाकि हमें $I_{n, m}$ और $I_{n, m-2}$ के मध्य सम्बन्ध स्थापित करना है, तो हम लिखते है

$P(x) = \sin^{n+1}x \cos^{m-1}x$ ....(1)

($I_{n,m}$ और $I_{n, m-2}$ में cosx की घात क्रमशः m और m – 2 है। दोनों में से न्यूनतम m – 2 है, न्यूनतम में 1 जोड़कर हम m – 2 + 1 = m – 1 प्राप्त करते हैं। अब P(x) में cos x की घात m – 1 लेते है। इसी प्रकार P(x) के लिए sin x की घात लिखते हैं।)

समीकरण (1) के दोनों पक्षों का अवकलन करने पर,

$P'(x) = (n+1)\sin^n x \cos^m x - (m-1)\sin^{n+2}x \cos^{m-2}x$
$= (n+1)\sin^n x \cos^m x - (m-1)\sin^n x (1-\cos^2 x)\cos^{m-2}x$
$= (n+1)\sin^n x \cos^m x - (m-1)\sin^n x \cos^{m-2} x + (m-1)\sin^n x \cos^m x$

$= (n + m) \sin^n x \cos^m x - (m - 1) \sin^n x \cos^{m-2} x$

अब दोनों पक्षों का समाकलन करने पर,

$\sin^{n+1} x \cos^{m-1} x = (n + m)\, I_{n,m} - (m - 1)\, I_{n,m-2}$

इसी प्रकार, हम अन्य सम्बन्ध स्थापित कर सकते हैं।

1. The relation between $I_{4,2}$ and $I_{2,2}$ is

$I_{4,2}$ और $I_{2,2}$ के मध्य सम्बन्ध है–

(A*) $I_{4,2} = \frac{1}{6}(-\sin^3 x \cos^3 x + 3I_{2,2})$ (B) $I_{4,2} = \frac{1}{6}(\sin^3 x \cos^3 x + 3I_{2,2})$

(C) $I_{4,2} = \frac{1}{6}(\sin^3 x \cos^3 x - 3I_{2,2})$ (D) $I_{4,2} = \frac{1}{6}(-\sin^3 x \cos^3 x + 2I_{2,2})$

Sol. Let (मानाकि) $P = \sin^3 x \cos^3 x$

$\frac{dp}{dx} = 3 \sin^2 x \cos^4 x - 3 \sin^4 x \cos^2 x = 3 \sin^2 x (1 - \sin^2 x) \cos^2 x - 3 \sin^4 x \cos^2 x$

$= 3 \sin^2 x \cos^2 x - 6 \sin^4 x \cos^2 x$

$\therefore \quad P = 3 I_{2,2} - 6 I_{4,2} \therefore \quad I_{4,2} = \frac{1}{6}(-P + 3 I_{2,2})$

2. The relation between $I_{4,2}$ and $I_{6,2}$ is

$I_{4,2}$ और $I_{6,2}$ के मध्य सम्बन्ध है–

(A*) $I_{4,2} = \frac{1}{5}(\sin^5 x \cos^3 x + 8I_{6,2})$ (B) $I_{4,2} = \frac{1}{5}(-\sin^5 x \cos^3 x + 8I_{6,2})$

(C) $I_{4,2} = \frac{1}{5}(\sin^5 x \cos^3 x - 8I_{6,2})$ (D) $I_{4,2} = \frac{1}{5}(\sin^5 x \cos^3 x + 8I_{6,2})$

Sol. Let (मानाकि) $P = \sin^5 x \cos^3 x$

$\therefore \quad \frac{dp}{dx} = 5 \sin^4 x \cos^4 x - 3 \sin^6 x \cos^2 x$

$= 5 \sin^4 x (1 - \sin^2 x) \cos^2 x - 3 \sin^6 x \cos^2 x = 5 \sin^4 x \cos^2 x - 8\sin^6 x \cos^2 x$

$\therefore \quad P = 5 I_{4,2} - 8I_{6,2} \therefore \quad I_{4,2} = \frac{1}{5}(P + 8I_{6,2})$

3. The relation between $I_{4,2}$ and $I_{4,4}$ is

$I_{4,2}$ और $I_{4,4}$ के मध्य सम्बन्ध है–

(A) $I_{4,2} = \frac{1}{3}(\sin^5 x \cos^3 x + 8 I_{4,4})$ (B*) $I_{4,2} = \frac{1}{3}(-\sin^5 x \cos^3 x + 8 I_{4,4})$

(C) $I_{4,2} = \frac{1}{3}(\sin^5 x \cos^3 x - 8 I_{4,4})$ (D) $I_{4,2} = \frac{1}{3}(\sin^5 x \cos^3 x + 6 I_{4,4})$

Sol. Let (मानाकि) $P = \sin^5 x \cos^3 x$

$\therefore \quad \frac{dp}{dx} = 5 \sin^4 x \cos^4 x - 3 \sin^6 x \cos^2 x$

$= 5 \sin^4 x \cos^4 x - 3 \sin^4 x (1 - \cos^2 x) \cos^2 x$

$= 8 \sin^4 x \cos^4 x - 3\sin^4 x \cos^2 x$

$\therefore \quad P = 8 I_{4,4} - 3 I_{4,2}$

$\therefore \quad I_{4,2} = \frac{1}{3}(-P + 8 I_{4,4})$

Comprehension # 2 (Q. No. 4 to 6)

It is known that

$$\sqrt{\tan x} + \sqrt{\cot x} = \begin{cases} \dfrac{\sqrt{\sin x}}{\sqrt{\cos x}} + \dfrac{\sqrt{\cos x}}{\sqrt{\sin x}} & \text{if} \quad 0 < x < \dfrac{\pi}{2} \\ \dfrac{\sqrt{-\sin x}}{\sqrt{-\cos x}} + \dfrac{\sqrt{-\cos x}}{\sqrt{-\sin x}} & \text{if} \quad \pi < x < \dfrac{3\pi}{2} \end{cases},$$

$$\frac{d}{dx}\left(\sqrt{\tan x} - \sqrt{\cot x}\right) = \frac{1}{2}\left(\sqrt{\tan x} + \sqrt{\cot x}\right)(\tan x + \cot x),\ \forall\ x \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$$

and $\dfrac{d}{dx}\left(\sqrt{\tan x} + \sqrt{\cot x}\right) = \dfrac{1}{2}\left(\sqrt{\tan x} - \sqrt{\cot x}\right)(\tan x + \cot x),\ \forall\ x \in \left(0, \dfrac{\pi}{2}\right) \cup \left(\pi, \dfrac{3\pi}{2}\right)$.

अनुच्छेद # 2 (प्रश्न संख्या 4 से 6)

यह ज्ञात है कि

$$\sqrt{\tan x} + \sqrt{\cot x} = \begin{cases} \dfrac{\sqrt{\sin x}}{\sqrt{\cos x}} + \dfrac{\sqrt{\cos x}}{\sqrt{\sin x}} & \text{यदि} \quad 0 < x < \dfrac{\pi}{2} \\ \dfrac{\sqrt{-\sin x}}{\sqrt{-\cos x}} + \dfrac{\sqrt{-\cos x}}{\sqrt{-\sin x}} & \text{यदि} \quad \pi < x < \dfrac{3\pi}{2} \end{cases},$$

$$\frac{d}{dx}\left(\sqrt{\tan x} - \sqrt{\cot x}\right) = \frac{1}{2}\left(\sqrt{\tan x} + \sqrt{\cot x}\right)(\tan x + \cot x),\ \forall\ x \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$$

और $\dfrac{d}{dx}\left(\sqrt{\tan x} + \sqrt{\cot x}\right) = \dfrac{1}{2}\left(\sqrt{\tan x} - \sqrt{\cot x}\right)(\tan x + \cot x),\ \forall\ x \in \left(0, \dfrac{\pi}{2}\right) \cup \left(\pi, \dfrac{3\pi}{2}\right)$.

तो निम्नलिखित प्रश्नों के उत्तर दीजिए –

4. Value of integral $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\ dx$, where $x \in \left(0, \dfrac{\pi}{2}\right) \cup \left(\pi, \dfrac{3\pi}{2}\right)$ is **[16JM120511]**

समाकल $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\ dx$, जहाँ $x \in \left(0, \dfrac{\pi}{2}\right) \cup \left(\pi, \dfrac{3\pi}{2}\right)$, का मान है –

(A*) $\sqrt{2}\ \tan^{-1}\left(\dfrac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + C$ (B) $\sqrt{2}\ \tan^{-1}\left(\dfrac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{2}}\right) + C$

(C) $-\sqrt{2}\ \tan^{-1}\left(\dfrac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + C$ (D) $-\sqrt{2}\ \tan^{-1}\left(\dfrac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{2}}\right) + C$

Sol. $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\ dx = 2\int \dfrac{(\sqrt{\tan x} + \sqrt{\cot x})\ (\tan x + \cot x)}{2(\tan x + \cot x)}\ dx$

$I = 2\int \dfrac{d\ (\sqrt{\tan x} - \sqrt{\cot x})}{(\sqrt{\tan x} - \sqrt{\cot x})^2 + 2} = \sqrt{2}\ \tan^{-1}\left(\dfrac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + C$

5. Value of the integral $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\ dx$, where $x \in \left(0, \dfrac{\pi}{2}\right)$, is **[16JM120512]**

समाकल $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\ dx$, जहाँ $x \in \left(0, \dfrac{\pi}{2}\right)$, का मान है –

(A) $\sqrt{2}\ \sin^{-1}(\cos x - \sin x) + C$ (B*) $\sqrt{2}\ \sin^{-1}(\sin x - \cos x) + C$

(C) $\sqrt{2}\ \sin^{-1}(\sin x + \cos x) + C$ (D) $-\sqrt{2}\ \sin^{-1}(\sin x + \cos x) + C$

Sol. If यदि $x \in \left(0\ ,\ \frac{\pi}{2}\right)$, then तब $I = \int(\sqrt{\tan x} + \sqrt{\cot x})\ dx = \int\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right) dx$

$= \int \frac{\sin x + \cos x}{\sqrt{\sin x\ .\ \cos x}}\ dx = \int \frac{\sqrt{2}\ d\ (\sin x - \cos x)}{\sqrt{1-(\sin x - \cos x)^2}}$

$= \sqrt{2}\ \sin^{-1}(\sin x - \cos x) + C$

6. Value of the integral $I = \int(\sqrt{\tan x} + \sqrt{\cot x})\ dx$, where $x \in \left(\pi\ ,\ \frac{3\pi}{2}\right)$, is **[16JM120513]**

समाकल $I = \int(\sqrt{\tan x} + \sqrt{\cot x})\ dx$, जहाँ $x \in \left(\pi\ ,\ \frac{3\pi}{2}\right)$, का मान है –

(A*) $\sqrt{2}\sin^{-1}(\cos x - \sin x) + C$
(B) $\sqrt{2}\sin^{-1}(\sin x - \cos x) + C$
(C) $\sqrt{2}\sin^{-1}(\sin x + \cos x) + C$
(D) $-\sqrt{2}\sin^{-1}(\sin x + \cos x) + C$

Sol. If यदि $x \in \left(\pi\ ,\ \frac{3\pi}{2}\right)$, then तब $I = \int(\sqrt{\tan x} + \sqrt{\cot x})dx = \int\left(\frac{\sqrt{-\sin x}}{\sqrt{-\cos x}} + \frac{\sqrt{-\cos x}}{\sqrt{-\sin x}}\right)dx$

$= -\int \frac{\sin x + \cos x}{\sqrt{\sin x\ .\ \cos x}}dx = -\int \frac{\sqrt{2}\ d\ (\sin x - \cos x)}{\sqrt{1-(\sin x - \cos x)^2}} = \sin^{-1}(\cos x - \sin x) + C$

Exercise-3

✍ **Marked questions are recommended for Revision.**
✍ **चिन्हित प्रश्न दोहराने योग्य प्रश्न है।**
*** Marked Questions may have more than one correct option.**
*** चिन्हित प्रश्न एक से अधिक सही विकल्प वाले प्रश्न है -**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

भाग - I : JEE (ADVANCED) / IIT-JEE (पिछले वर्षो) के प्रश्न

1. Let $f(x) = \int e^x (x-1)(x-2)\,dx$ then f decreases in the interval: **[IIT-JEE 2000, Scr, (1, 0), 35]**

यदि $f(x) = \int e^x (x-1)(x-2)\,dx$ हो, तो f जिस अन्तराल में ह्रासमान है–

(A) $(-\infty, 2)$ (B) $(-2, -1)$ (C*) $(1, 2)$ (D) $(2, +\infty)$

Solution :

$\because$ $f(x) = \int e^x(x-1)(x-2)\,dx$

$\Rightarrow$ $f'(x) = e^x(x-1)(x-2) \le 0$ $\quad 1 \le x \le 2$

$\Rightarrow$ $x \in [1, 2]$

2. Evaluate, $\int \sin^{-1}\left(\dfrac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$. **[IIT-JEE 2001, Main, (5, 0), 100]**

$\int \sin^{-1}\left(\dfrac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$ का मान ज्ञात कीजिए।

Ans. $(x+1)\tan^{-1}\left(\dfrac{2x+2}{3}\right) - \dfrac{3}{4}\ \ell n\,(4x^2 + 8x + 13) + C$

Sol. $I = \int \sin^{-1}\left(\dfrac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$

$= \int \sin^{-1}\left(\dfrac{2x+2}{\sqrt{(2x+2)^2+9}}\right) dx$

Put $2x + 2 = 3\tan\theta$ (रखने पर) $\Rightarrow$ $2dx = 3\sec^2\theta\, d\theta$

$\Rightarrow$ $I = \int \sin^{-1}\left(\dfrac{3\tan\theta}{\sqrt{9\tan^2\theta+9}}\right).\dfrac{3}{2}\sec^2\theta\ \ d\theta$

$= \dfrac{3}{2}\int \sin^{-1}(\sin\theta).|\sec^2\theta\ |\ d\theta$

$= \dfrac{3}{2}\int \theta.\sec^2\theta\ \ d\theta = \dfrac{3}{2}[\theta.\tan\theta - \int 1.\tan\theta d\theta]$

$= \dfrac{3}{2}[\theta.\tan\theta - \ell n\ \ \sec\theta] + C$

$= \dfrac{3}{2}\left[\tan^{-1}\left(\dfrac{2x+2}{3}\right)\ .\left(\dfrac{2x+2}{3}\right) - \ell n\ \sqrt{1+\left(\dfrac{2x+2}{3}\right)^2}\ \right] + C_1$

$= (x + 1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\ \ell n\ (4x^2 + 8x + 13) + C$

3. For any natural number m, evaluate,

$\int \left(x^{3m} + x^{2m} + x^m\right)\left(2x^{2m} + 3x^m + 6\right)^{1/m} dx,\ x > 0.$ **[IIT-JEE 2002, Main, (5, 0), 60]**

किसी प्राकृत संख्या m के लिए

$\int \left(x^{3m} + x^{2m} + x^m\right)\left(2x^{2m} + 3x^m + 6\right)^{1/m}$ का मान ज्ञात कीजिए।

Ans. $\frac{(2x^{3m} + 3x^{2m} + 6x^m)^{\frac{m+1}{m}}}{6(m+1)} + C$

Sol. For any natural number m, the given integral can be written as,
(किसी प्राकृत संख्या m के लिए, दिया गया समाकलन लिखा जा सकता है)

$I = \int \left(x^{3m} + x^{2m} + x^m\right)\frac{(2x^{3m} + 3x^{2m} + 6x^m)^{1/m}}{x}\ dx$

$= \int (2x^{3m} + 3x^{2m} + 6x^m)^{1/m}\ (x^{3m-1} + x^{2m-1} + x^{m-1})\ dx$

put $2x^{3m} + 3x^{2m} + 6x^m = t$ रखने पर $\Rightarrow$ $(x^{3m-1} + x^{2m-1} + x^{m-1})\ dx = dt$

$\Rightarrow$ $I = \int t^{1/m}.\frac{dt}{6m}$, where (जहाँ) $t = 2x^{3m} + 3x^{2m} + 6x^m$

$= \frac{1}{6m}.\frac{t^{\frac{1}{m}+1}}{\left(\frac{1}{m}+1\right)} + C$

$= \frac{1}{6(m+1)}.(2x^{3m} + 3x^{2m} + 6x^m)^{(m+1)/m} + C$

4. $\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}}\ dx$ is equal to **[IIT-JEE 2006, (3, –1), 184]**

$\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}}\ dx$ का मान है–

(A) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + C$
(B) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^3} + C$
(C) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x} + C$
(D*) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$

Sol. $I = \int \frac{\left(\frac{1}{x^3} - \frac{1}{x^5}\right)}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}\ dx$ put $2 - \frac{2}{x^2} + \frac{1}{x^4} = z$ रखने पर $\Rightarrow$ $\left(\frac{4}{x^3} - \frac{4}{x^5}\right) dx = dz$

$\Rightarrow I = \frac{1}{4}\int \frac{dz}{\sqrt{z}}$

$= \frac{1}{2}\sqrt{z} + c \Rightarrow \frac{1}{2}\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}} + C = \frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$

5. Let $f(x) = \frac{x}{(1 + x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = \underbrace{(f \circ f \circ .... \circ f)}_{f\ \text{occurs}\ n\ \text{times}}(x)$. Then $\int x^{n-2} g(x)\ dx$ equals

[IIT-JEE 2007, Paper-2, (3, – 1), 81]

माना कि $n \geq 2$ के लिए $f(x) = \frac{x}{(1+x^n)^{1/n}}$ तथा $g(x) = \underbrace{(f \circ f \circ .... \circ f)}_{\text{f occurs n times}}(x)$ हो, तो $\int x^{n-2} g(x)$ का मान है–

(A*) $\frac{1}{n(n-1)}\left(1+nx^n\right)^{1-\frac{1}{n}} + K$ (B) $\frac{1}{(n-1)}\left(1+nx^n\right)^{1-\frac{1}{n}} + K$

(C) $\frac{1}{n(n+1)}\left(1+nx^n\right)^{1+\frac{1}{n}} + K$ (D) $\frac{1}{(n+1)}\left(1+nx^n\right)^{1+\frac{1}{n}} + K$

Sol. $f(f(x)) = \frac{f(x)}{(1+(f(x)^n)^{1/n}}$

$= \frac{x/(1+x^n)^{1/n}}{\left(1+\frac{x^n}{1+x^n}\right)^{1/n}} = \frac{x}{(1+2x^n)^{1/n}}$

Also, तथा $f(f(f(x))) = \frac{x/(1+2x^n)^{1/n}}{\left(1+\frac{x^n}{1+2x^n}\right)^{1/n}} = \frac{x}{\left(1+3x^n\right)^{1/n}}$ $\Rightarrow$ $g(x) = \frac{x}{(1+nx^n)^{1/n}}$

Hence अतः $I = \int x^{n-2} \frac{x}{(1+nx^n)^{1/n}} dx = \int \frac{x^{n-1}}{(1+nx^n)^{1/n}} dx$ Let (माना कि) $1 + n x^n = t \Rightarrow x^{n-1} dx = \frac{1}{n^2}$ dt

$\Rightarrow$ $I = \frac{1}{n^2}\int \frac{dt}{t^{1/n}} = \frac{1}{n^2} \frac{t^{\frac{-1}{n}+1}}{1-\frac{1}{n}} = \frac{1}{n(n-1)} (1 + nx^n)^{1-1/n} + K$

6. Let F(x) be an indefinite integral of $\sin^2 x$. **[IIT-JEE 2007, Paper-1, (3, – 1), 81]**
STATEMENT-1 : The function F(x) satisfies $F(x + \pi) = F(x)$ for all real x.
because
STATEMENT-2 : $\sin^2(x + \pi) = \sin^2 x$ for all real x.
(A) Statement-1 is True, Statement-2 is True ; Statement-2 **is** a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True ; Statement-2 **is NOT** a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D*) Statement-1 is False, Statement-2 is True
माना कि F(x), $\sin^2 x$ का अनिश्चित समाकल (indefinite integral) है।
कथन 1 : फलन (function) F(x), सभी वास्तविक (real) x के लिए $F(x + \pi) = F(x)$ को संतुष्ट करता है।
क्योंकि
कथन 2 : सभी वास्तविक (real) x के लिए $\sin^2 (x + \pi) = \sin^2 x$
(A) कथन - 1 सत्य है, कथन - 2 सत्य है; कथन - 2 कथन - 1 का सही स्पष्टीकरण हैं
(B) कथन - 1 सत्य है, कथन - 2 सत्य है; कथन - 2 कथन - 1 का सही स्पष्टीकरण नहीं हैं
(C) कथन - 1 सत्य है, कथन - 2 असत्य है
(D) कथन - 1 असत्य है, कथन - 2 सत्य ह

Solution

Statement - 1 $F(x) = \int (\sin^2 x)\ dx = \int \frac{1-\cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$
Which is aperiodic function. Hence statement is false.
Statement - 2 $\sin^2 x$ is periodic with the period π, hence statement is true.

Hindi.

कथन - 1 $F(x) = \int (\sin^2 x)\ dx = \int \frac{1-\cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$
जो कि अनावर्ती फलन है। इस प्रकार कथन असत्य है।
कथन - 2 चूँकि $\sin^2 x$ फलन π आवर्त का आवर्ती फलन है, अतः सत्य है।

7. Let $I = \int \frac{e^x}{e^{4x}+e^{2x}+1}\,dx$, $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}\,dx$. Then, for an arbitrary constant C, the value of J – I is equal to : **[IIT-JEE 2008, Paper-2, (3, – 1), 81]**

माना $I = \int \frac{e^x}{e^{4x}+e^{2x}+1}\,dx$, $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}\,dx$. तब एक स्वेच्छ अचर (arbitrary constant) C के लिए J – I का मान है –

(A) $\frac{1}{2}\ell n\left|\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right| + C$ (B) $\frac{1}{2}\ell n\left|\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right| + C$

(C*) $\frac{1}{2}\ell n\left|\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right| + C$ (D) $\frac{1}{2}\ell n\left|\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right| + C$

Sol. $J - I = \int \frac{(e^{3x}-e^{x})\ \ dx}{e^{4x}+e^{2x}+1}$

Let (माना) $e^x = t$ $\Rightarrow$ $e^x\,dx = dt$

$\Rightarrow$ $J - I = \int \frac{t^2-1}{t^4+t^2+1}\,dt = \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,dt$ put $t+\frac{1}{t} = u$ रखने पर $\Rightarrow \left(1-\frac{1}{t^2}\right)\,dt = du$

$\Rightarrow$ $J - I = \int \frac{du}{u^2-1} = \frac{1}{2}\ \ell n\left|\frac{u-1}{u+1}\right| + C = \frac{1}{2}\ \ell n\left|\frac{t^2-t+1}{t^2+t+1}\right| + C = \frac{1}{2}\ \ell n\left|\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right| + C$

8. The integral $\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$ equals (for some arbitrary constant K)

समाकलन $\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$ का मान निम्न है (किसी यादृच्छिक अचर (arbitrary constant) K के लिये)

(A) $\frac{-1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\} + K$ **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

(B) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\} + K$

(C*) $\frac{-1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\} + K$

(D) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\} + K$

Sol. **(C)**

Put secx + tanx = t

$(\sec x \tan x + \sec^2 x)\,dx = dt$

$\sec x\ .\ t\,dx = dt$

$\sec x - \tan x = \frac{1}{t}$

$\sec x = \frac{t+\frac{1}{t}}{2}$

$\int \frac{\sec x.dt}{t^{9/2}.t} = \int \frac{1}{2}\frac{\left(t+\frac{1}{t}\right)}{t\ \ .\ \ t^{9/2}}dt$

$= \frac{1}{2}\int\left(\frac{1}{t^{9/2}}+\frac{1}{t^{13/2}}\right)\ dt$

$$= -\frac{1}{2}\left[\frac{2}{7t^{7/2}} + \frac{2}{11t^{11/2}}\right] + k$$

$$= -\frac{1}{t^{11/2}}\left[\frac{t^2}{7} + \frac{1}{11}\right] + k$$

Hindi Ans (C)

secx + tanx = t रखने पर

(secx tanx + sec^2x) dx = dt

secx . t dx = dt

secx – tanx = $\frac{1}{t}$

secx = $\dfrac{t + \frac{1}{t}}{2}$

$$\int \frac{\sec x.dt}{t^{9/2}.t} = \int \frac{1}{2} \frac{\left(t + \frac{1}{t}\right)}{t \;.\; t^{9/2}} dt$$

$$= \frac{1}{2}\int \left(\frac{1}{t^{9/2}} + \frac{1}{t^{13/2}}\right) dt$$

$$= -\frac{1}{2}\left[\frac{2}{7t^{7/2}} + \frac{2}{11t^{11/2}}\right] + k$$

$$= -\frac{1}{t^{11/2}}\left[\frac{t^2}{7} + \frac{1}{11}\right] + k$$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

भाग - II : JEE (MAIN) / AIEEE (पिछले वर्षो) के प्रश्न

1. If the integral $\int \frac{5\tan x}{\tan x - 2}$ dx = x + a ℓn |sin x – 2 cos x| + k, then a is equal to : **[AIEEE-2012, (4, –1)/120]**

यदि समाकलन $\int \frac{5\tan x}{\tan x - 2}$ dx = x + a ℓn |sin x – 2 cos x| + k है, तो a बराबर है– **[AIEEE-2012, (4, –1)/120]**

(1) – 1 (2) – 2 (3) 1 (4*) 2

Sol. Ans. (4)

$$\int \frac{5\tan x}{\tan x - 2} dx$$

$$= \int \frac{5\sin x}{\sin x - 2\cos x} dx$$

$$= \int \frac{(\sin x - 2\cos x) + 2(\cos x + 2\sin x)}{(\sin x - 2\cos x)} dx$$

$$= \int dx + 2\int \frac{\cos x + 2\sin x}{\sin x - 2\cos x} dx$$

= x + 2ℓn |(sin x – 2 cos x)| + k

$\Rightarrow$ a = 2

2. If $\int f(x)\ dx = \psi(x)$, then $\int x^5 f(x^3)\ dx$ is equal to **[AIEEE - 2013, (4, –1),360]**

यदि $\int f(x)\ dx = \psi(x)$ है, तो $\int x^5 f(x^3)\ dx$ बराबर है– **[AIEEE - 2013, (4, –1),360]**

(1) $\frac{1}{3}\left[x^3\psi(x^3) - \int x^2\psi(x^3)\ dx\right] + C$ (2) $\frac{1}{3}x^3\psi(x^3) - 3\int x^3\psi(x^3)\ dx + C$

(3*) $\frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)\ dx + C$ (4) $\frac{1}{3}\left[x^3\psi(x^3) - \int x^3\psi(x^3)\ dx\right] + C$

Sol. (3)

$\int f(x)dx = \psi\ (x)$

$I = \int x^5 f(x^3)dx$

put $x^3 = t$ रखने पर $\Rightarrow$ $x^2dx = \frac{dt}{3}$

$= \frac{1}{3}\int t f(t)dt$

$= \frac{1}{3}\left[t\psi(t) - \int \psi(t)dt\right]$

$= \frac{1}{3}\left[x^3\psi(x^3) - 3\int x^2\psi(x^3)dx\right] + c$

$= \frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)dx + c$

3. The integral $\int\left(1 + x - \frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ is equal to : **[Indefinite Integration] [JEE(Main) 2014, (4, – 1), 120]**

समाकल $\int\left(1 + x - \frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ बराबर है : **[Indefinite Integration] [JEE(Main) 2014, (4, – 1), 120]**

(1) $(x + 1)\ e^{x+\frac{1}{x}} + c$ (2) $-x\,e^{x+\frac{1}{x}} + c$ (3) $(x - 1)\ e^{x+\frac{1}{x}} + c$ (4) $x\ e^{x+\frac{1}{x}} + c$

Sol. **Ans. (4)**

$\int \left(1 + x - \frac{1}{x}\right)e^{\left(x+\frac{1}{x}\right)}dx$

$\int e^{x+\frac{1}{x}}dx + \int \left(x - \frac{1}{x}\right) e^{x+\frac{1}{x}}dx$

Applying integration by parts in (I) (I) मे समाकलन करने पर

We get,

$$\int e^{x+\frac{1}{x}}dx = x.\ e^{x+\frac{1}{x}} - \int x\left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}}dx = x\ e^{x+\frac{1}{x}} - \int \left(x - \frac{1}{x}\right) e^{x+\frac{1}{x}}dx$$

Thus अतः $\int e^{x+\frac{1}{x}}dx + \int \left(x - \frac{1}{x}\right) e^{x+\frac{1}{x}}dx = x\ e^{x+\frac{1}{x}} + C.$

4. The integral $\int \frac{dx}{x^2\left(x^4+1\right)^{3/4}}$ equals **[JEE(Main) 2015, (4, – 1), 120]**

(1) $\left(\frac{x^4+1}{x^4}\right)^{1/4} + c$ (2) $(x^4 + 1)^{1/4} + c$ (3) $-(x^4 + 1)^{1/4} + c$ (4) $-\left(\frac{x^4+1}{x^4}\right)^{1/4} + c$

समाकल $\int \frac{dx}{x^2\left(x^4+1\right)^{3/4}}$ बराबर है– **[JEE(Main) 2015, (4, –1), 120]**

(1) $\left(\frac{x^4+1}{x^4}\right)^{1/4} + c$ (2) $(x^4 + 1)^{1/4} + c$ (3) $-(x^4 + 1)^{1/4} + c$ (4) $\left(\frac{x^4+1}{x^4}\right)^{1/4} -+ c$

Ans. **(4)**

Sol. $\int \frac{dx}{x^2(x^4+1)^{3/4}}$

$\int \frac{dx}{x^3\left(1+\frac{1}{x^4}\right)^{3/4}}$

$1 + \frac{1}{x^4} = t^4$

$-4\frac{1}{x^5}\,dx = 4t^3\,dt$

$\frac{dx}{x^3} = -t^3\,dt$

$\int \frac{-t^3dt}{t^3} = -t + C = -\left(1+\frac{1}{x^4}\right)^{1/4} + C$

5. The integral $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$ is equal to **[JEE(Main) 2016, (4, –1), 120]**

(1) $\frac{x^{10}}{2(x^5+x^3+1)^2}+C$ (2) $\frac{x^5}{2(x^5+x^3+1)^2}+C$ (3) $\frac{-x^{10}}{2(x^5+x^3+1)^2}+C$ (4) $\frac{-x^5}{(x^5+x^3+1)^2}+C$

where C is an arbitrary constant

समाकल $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$ बराबर है–

(1) $\frac{x^{10}}{2(x^5+x^3+1)^2}+C$ (2) $\frac{x^5}{2(x^5+x^3+1)^2}+C$ (3) $\frac{-x^{10}}{2(x^5+x^3+1)^2}+C$ (4) $\frac{-x^5}{(x^5+x^3+1)^2}+C$

जहाँ C एक स्वेच्छ अचर है।

Ans. **(1)**

Sol. $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$

$\int \frac{\left(\frac{2}{x^3}+\frac{5}{x^6}\right)}{\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^3}dx$

Let (माना) $1+\frac{1}{x^2}+\frac{1}{x^5}=t$

$$\frac{dt}{dx}=\frac{-2}{x^3}-\frac{5}{x^6}$$

$$\int\frac{-dt}{t^3}=\frac{1}{2t^2}+C=\frac{1}{2\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^2}+C=\frac{x^{10}}{2(x^5+x^3+1)^2}+C$$

6. Let $I_n=\int\tan^n x\,dx$, $(n>1)$. If $I_4+I_6=a\tan^5x+bx^5+C$, where C is a constant of integration, then the ordered pair (a, b) is equal to **[JEE(Main) 2017, (4, –1), 120]**

माना $I_n=\int\tan^n x\,dx$, $(n>1)$ है। यदि $I_4+I_6=a\tan^5x+bx^5+C$ है, जहाँ C एक समाकलन अचर है तो क्रमित युग्म (a, b) बराबर है : **[JEE(Main) 2018, (4, –1), 120]**

(1) $\left(-\frac{1}{5},1\right)$ (2*) $\left(\frac{1}{5},0\right)$ (3) $\left(\frac{1}{5},-1\right)$ (4) $\left(-\frac{1}{5},0\right)$

Ans. **(2)**

Sol. $I_n=\int\tan^n x\,dx=\int\tan^{n-2}(\sec^2x-1)dx$

$$\int(\tan x)^{n-2}\sec^2x\,dx-\int(\tan x)^{n-2}dx=\frac{(\tan x)^{n-1}}{n-1}-I_{n-2}$$

$$I_n+I_{n-2}=\frac{(\tan x)^{n-1}}{n-1}$$

put n = 6 रखने पर

$$I_4+I_6=\frac{1}{5}\tan^5x=a\tan^5x+bx^5+c\quad\Rightarrow\quad a=\frac{1}{5}\quad b=0\quad c=0$$

$\therefore$ $(a,b)=\left(\frac{1}{5},0\right)$

7. The integral $\int\frac{\sin^2x\cos^2x}{(\sin^5x+\cos^3x\sin^2x+\sin^3x\cos^2x+\cos^5x)^2}dx$ is equal to : **[JEE(Main) 2018, (4, –1), 120]**

समाकलन $\int\frac{\sin^2x\cos^2x}{(\sin^5x+\cos^3x\sin^2x+\sin^3x\cos^2x+\cos^5x)^2}dx$ बराबर है :

(1) $\frac{1}{1+\cot^3x}+C$ (2) $\frac{-1}{1+\cot^3x}+C$ (3) $\frac{1}{3(1+\tan^3x)}+C$ (4*) $\frac{-1}{3(1+\tan^3x)}+C$

(where C is a constant of integration)

(जहाँ C एक समाकलन अचर है)

Sol. **(4)**

$$I = \int \frac{\tan^2 x.\sec^2 x}{(\tan^5 x + \tan^2 x + \tan^3 x + 1)^2}\,dx = \int \frac{\tan^2 x \sec^6 x}{(\tan^2 x + 1)^2(\tan^3 x + 1)^2}dx = \int \frac{\tan^2 x \sec^2 x}{(1+\tan^3 x)^2}dx$$

let 1 + $\tan^3 x$ = t

$3\tan^2 x \sec^2 x\, dx = dt$ = $\frac{1}{3}\int \frac{1}{t^2}dt = -\frac{1}{3(1+\tan^3 x)} + C$

8. Let $n \geq 2$ be a natural number and $0 < \theta < \pi / 2$. Then $\int \frac{(\sin^n \theta - \sin\theta)^{\frac{1}{n}} \cos\theta}{\sin^{n+1}\theta}\, d\theta$ is equal to :

(where C is a constant of integration) **[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]**

माना $n \geq 2$ एक प्राकृत संख्या है तथा $0 < \theta < \pi / 2$ है, तो $\int \frac{(\sin^n \theta - \sin\theta)^{\frac{1}{n}} \cos\theta}{\sin^{n+1}\theta}\, d\theta$ बराबर है:

(जहाँ C एक समाकलन अचर है)

(1) $\frac{n}{n^2-1}\left(1 - \frac{1}{\sin^{n+1}\theta}\right)^{\frac{n+1}{n}} + C$

(2) $\frac{n}{n^2-1}\left(1 - \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

(3) $\frac{n}{n^2+1}\left(1 - \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

(4) $\frac{n}{n^2-1}\left(1 + \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

Ans. **(2)**

Sol. $\int \frac{\left(\sin^n \theta - \sin\theta\right)^{\frac{1}{n}} \cos\theta}{\sin^{n+1}\theta}d\theta$

$= \int \frac{\left(t^n - t\right)^{\frac{1}{n}} dt}{t^{n+1}}$ (Put $\sin\theta = t$, रखने पर)

$= \int \frac{t\left(1 - \frac{1}{t^{n-1}}\right)^{\frac{1}{n}}}{t^{n+1}}dt$

$= \int \frac{\left(1 - \frac{1}{t^{n-1}}\right)^{\frac{1}{n}}}{t^{n}}dt$

Put $1-\frac{1}{t^{n-1}}=z \Rightarrow \frac{(n-1)}{t^n}dt=dz$, रखने पर

$$\Rightarrow I = \frac{1}{n-1}\int z^{\frac{1}{n}}dz = \frac{z^{\frac{1}{n}+1}}{\left(\frac{1}{n}+1\right)(n-1)}+c = \frac{n\left(1-t^{1-n}\right)^{\frac{1}{n}+1}}{n^2-1}+c$$

$$= \frac{n}{n^2-1}\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+c$$

9. The integral $\int \cos(\log_e x)dx$ is equal to : (where C is a constant of integration

समाकल $\int \cos(\log_e x)dx$ बराबर है– (जहाँ C एक समाकलन अचर है)

[JEE(Main) 2019, Online (12-01-19),P-1 (4, – 1), 120]

(1) $x\left[\cos(\log_e x)-\sin(\log_e x)\right]+C$

(2) $\frac{x}{2}\left[\sin(\log_e x)-\cos(\log_e x)\right]+C$

(3) $x[\cos(\log_e x)+\sin(\log_e x)]+C$

(4) $\frac{x}{2}\left[\cos(\log_e x)+\sin(\log_e x)\right]+C$

Ans. **(4)**

Sol. By parts खण्डशः समाकलन से

$I = x\cos(\log x) + \int \frac{x}{x}\sin(\log x)dx$

$x\ \cos(\log x) + \int \sin(\log x)dx$

$I = x\cos(\log x) + x\sin(\log x) - \int \cos\log x dx + c$

$I = \frac{x}{2}(\cos(\log x)+\sin(\log x)) + c$

10. $\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}}dx$ is equal to : **[JEE(Main) 2019, Online (08-04-19),P-1 (4, – 1), 120] [Indefinite Integration]**

(where c is a constant of integration)

$\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}}dx$ बराबर है–

(जहाँ c एक समाकलन अचर है)

(1) $x + 2\sin x + 2\sin 2x + c$ (2) $2x + \sin x + \sin 2x + c$

(3) $2x + \sin x + 2\sin 2x + c$ (4) $x + 2\sin x + \sin 2x + c$

Ans. (4)

Sol. $\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}} dx = \int \frac{2\sin\frac{5x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx = \int \left(\frac{\sin 3x + \sin 2x}{\sin x}\right) dx = \int 2\cos x\, dx + \int (3 - 4\sin^2 x)\, dx$

$= 2\int \cos x\, dx + \int dx + 2\int \cos 2x\, dx = 2\sin x + x + \sin 2x + C$

11. If $\int e^{\sec x}(\sec x \tan x f(x) + (\sec x \tan x + \sec^2 x))dx = e^{\sec x} f(x) + C$, then a possible choice of f(x) is :

यदि $\int e^{\sec x}(\sec x \tan x f(x) + (\sec x \tan x + \sec^2 x))dx = e^{\sec x} f(x) + C$, तो f(x) का एक संभव विकल्प (choice) है–   **[JEE(Main) 2019, Online (09-04-19),P-2 (4, – 1), 120] [Indefinite Integration]**

(1) $\sec x + x\tan x - \frac{1}{2}$ (2) $\sec x + \tan x + \frac{1}{2}$

(3) $x\sec x + \tan x + \frac{1}{2}$ (4) $\sec x - \tan x - \frac{1}{2}$

Ans. (2)

Sol. $\int e^{\sec x} (\sec x \tan x\, f(x) + \sec x \tan x + \sec^2 x)\, dx = e^{\sec x} f(x)$

Differentiate both side

$e^{\sec x} (\sec x \tan x\, f(x) + \sec x \tan x + \sec^2 x) = e^{\sec x} f'(x) + e^{\sec x} \sec x \tan x\, f(x)$

$\therefore$ $\sec x \tan x + \sec^2 x = f'(x)$

$\therefore$ $f(x) = \sec x + \tan x + d$

12. If $\int \frac{dx}{(x^2 - 2x + 10)^2} = A\left(\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{f(x)}{x^2 - 2x + 10}\right) + C$

Where C is a constant of integration, then

(1) $A = \frac{1}{27}$ and $f(x) = 9(x - 1)$ (2) $A = \frac{1}{54}$ and $f(x) = 9(x - 1)^2$

(3) $A = \frac{1}{81}$ and $f(x) = 3(x - 1)$ (4) $A = \frac{1}{54}$ and $f(x) = 3(x - 1)$

यदि $\int \frac{dx}{(x^2-2x+10)^2} = A\left(\tan^{-1}\left(\frac{x-1}{3}\right)+\frac{f(x)}{x^2-2x+10}\right)+C$

जहाँ C एक समाकलन अचर है, तो :

(1) $A = \frac{1}{27}$ तथा $f(x) = 9(x-1)$ (2) $A = \frac{1}{54}$ तथा $f(x) = 9(x-1)^2$

(3) $A = \frac{1}{81}$ तथा $f(x) = 3(x-1)$ (4) $A = \frac{1}{54}$ तथा $f(x) = 3(x-1)$

Ans. (4) **[JEE(Main) 2019, Online (10-04-19),P-1 (4, – 1), 120] [Indefinite Integration]**

Sol. $\int \frac{dy}{(x^2-2x+10)^2} = \int \frac{dx}{((x-1)^2+9)^2}$

Put $x-1 = 3\tan\theta$

$dx = 3\sec^2\theta d\theta$

$\Rightarrow \int \frac{3\sec^2\theta d\theta}{(9\sec^2\theta)^2} = \frac{1}{27}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{27}\int \cos^2\theta d\theta$

$= \frac{1}{27}\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{1}{54}\left[\theta+\frac{1}{2}\sin 2\theta\right]+C = \frac{1}{54}\left[\tan^{-1}\left(\frac{x-1}{3}\right)+\frac{3(x-1)}{x^2-2x+10}\right]+C$

13. If $\int \frac{d\theta}{\cos^2\theta(\tan 2\theta+\sec 2\theta)} = \lambda\tan\theta + 2\log_e|f(\theta)| + C$ where C is a constant of integration, then the ordered pair $(\lambda, f(\theta))$ is equal to :

यदि $\int \frac{d\theta}{\cos^2\theta(\tan 2\theta+\sec 2\theta)} = \lambda\tan\theta + 2\log_e|f(\theta)| + C$ है, जहाँ C एक समाकलन अचर है, तो क्रमित युग्म $(\lambda, f(\theta))$ बराबर है: **[JEE(Main) 2020, Online (09-01-20),P-2 (4, –1), 120]**

(1) $(1, 1+\tan\theta)$ (2) $(-1, 1-\tan\theta)$

(3) $(-1, 1+\tan\theta)$ (4) $(1, 1-\tan\theta)$

Ans. (3)

Sol. $\int \frac{\sec^2\theta}{\frac{1+\tan^2\theta}{1-\tan^2\theta}+\frac{2\tan\theta}{1-\tan^2\theta}} d\theta$

$= \int \frac{\sec^2\theta(1-\tan^2\theta)}{(1+\tan\theta)^2} d\theta$

$= \int \frac{\sec^2\theta(1-\tan\theta)}{1+\tan\theta} d\theta$

$\tan\theta = t \Rightarrow \sec^2\theta\, d\theta = dt$

$= \int \left(\frac{1-t}{1+t}\right) dt = \int \left(-1+\frac{2}{1+t}\right)dt$

$= -t + 2\log(1+t) + C$

$= -\tan\theta + 2\log(1+\tan\theta) + C$

$\Rightarrow \quad \lambda = -1$ and $f(\theta) = 1 + \tan\theta$

High Level Problems (HLP)

1. Evaluate : $\int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}\, dx$

$\int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}\, dx$ को सरल कीजिए –

Ans. $-\frac{1}{2}\sin 2x + C$

Sol. $\int \frac{\sin^8 x - \cos^8 x}{\sin^4 x + \cos^4 x}\, dx = \int (\sin^4 x - \cos^4 x)\, dx = \int (\sin^2 x - \cos^2 x)\, dx = -\frac{1}{2}\sin 2x + C$

2. Evaluate : $\int \frac{\cos 5x + \cos 4x}{1-2\cos 3x}dx$

$\int \frac{\cos 5x + \cos 4x}{1-2\cos 3x}dx$ को सरल कीजिए –

Ans. $-\left(\sin x + \frac{\sin 2x}{2}\right) + C$

Sol. $\frac{\cos 5x + \cos 4x}{1-2\cos 3x}dx = \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}\, dx}{1-2\left(2\cos^2\left(\frac{3x}{2}\right)-1\right)}$

$= \int \frac{2\cos\frac{x}{2}\left(4\cos^3\frac{3x}{2} - 3\cos\frac{3x}{2}\right)}{3-4\cos^2\frac{3x}{2}}\, dx = -\int 2\cos\frac{3x}{2}\cos\frac{x}{2}\, dx = -\int \cos 2x\, dx - \int \cos x\, dx$

$= -\left(\frac{\sin 2x}{2} + \sin x\right) + C$

3. Evaluate : $\int \sqrt{x+\sqrt{x^2+2}}\, dx$

$\int \sqrt{x+\sqrt{x^2+2}}\, dx$ को सरल कीजिए – **Ans.** $\frac{1}{3}\left(x+\sqrt{x^2+2}\right)^{3/2} - \frac{2}{\left(x+\sqrt{x^2+2}\right)^{1/2}} + C$

Sol. $I = \int \sqrt{x+\sqrt{x^2+2}}\, dx$ Put $x+\sqrt{x^2+2} = t \Rightarrow x = \left(\frac{t^2-2}{2t}\right) \Rightarrow dx = \frac{t^2+2}{2t^2}\, dt$

$\Rightarrow \quad I = \int \sqrt{t}\ .\ \left(\frac{2+t^2}{2t^2}\right) dt = \int \left(\frac{1}{2}t^{1/2} + t^{-3/2}\right) dt = \frac{1}{2} \times \frac{2}{3}\ t^{3/2} - 2t^{-1/2} + C$

$= \frac{1}{3}\ (x+\sqrt{x^2+2})^{3/2} \ - 2\,(x+\sqrt{x^2+2})^{-1/2}\ + C$

Hindi $I = \int \sqrt{x+\sqrt{x^2+2}}\ dx$

माना $\quad x+\sqrt{x^2+2} = t \ \Rightarrow x = \left(\frac{t^2-2}{2t}\right) \ \Rightarrow dx = \frac{t^2+2}{2t^2}\ dt$

$\Rightarrow \quad I = \int \sqrt{t}\ .\ \left(\frac{2+t^2}{2t^2}\right) dt = \int \left(\frac{1}{2}t^{1/2} + t^{-3/2}\right) dt = \frac{1}{2} \times \frac{2}{3}\ t^{3/2} - 2t^{-1/2} + C$

$= \frac{1}{3}\ (x+\sqrt{x^2+2})^{3/2} \ - 2\,(x+\sqrt{x^2+2})^{-1/2}\ + C$

4. Evaluate : $\int \frac{dx}{(x^3+3x^2+3x+1)\sqrt{x^2+2x-3}}$ को सरल कीजिए –

Ans. $\frac{\sqrt{x^2+2x-3}}{8(x+1)^2} + \frac{1}{16}\,.\cos^{-1}\left(\frac{2}{x+1}\right) + C$

Sol. $I = \int \frac{dx}{(x+1)^3\sqrt{(x+1)^2-4}}$, $\quad$ put, x + 1 = 2 sec θ $\quad \Rightarrow \quad dx = 2\sec\theta\tan\theta\, d\theta$

$I = \int \frac{2\sec\theta\tan\theta}{8\sec^3\theta\ .\ 2\tan\theta}\ d\theta = \frac{1}{16}\int (1+\cos 2\theta)\ d\theta$

$= \frac{1}{16}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{1}{16}\left(\cos^{-1}\frac{2}{x+1} + \sqrt{1-\left(\frac{2}{x+1}\right)^2}\ \frac{2}{x+1}\right) + C$

$= \frac{1}{16}\ \cos^{-1}\ \frac{2}{x+1} + \frac{1}{8}\ \frac{\sqrt{x^2+2x-3}}{(x+1)^2} + C$

Hindi $I = \int \frac{dx}{(x+1)^3\sqrt{(x+1)^2-4}}$, $\quad$ माना, x + 1 = 2 sec θ $\quad \Rightarrow \quad dx = 2\sec\theta\tan\theta\, d\theta \quad I$

$= \int \frac{2\sec\theta\tan\theta}{8\sec^3\theta\ .\ 2\tan\theta}\ d\theta = \frac{1}{16}\int (1+\cos 2\theta)\ d\theta$

$= \frac{1}{16}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{1}{16}\left(\cos^{-1}\frac{2}{x+1} + \sqrt{1-\left(\frac{2}{x+1}\right)^2}\ \frac{2}{x+1}\right) + C$

$= \frac{1}{16}\ \cos^{-1}\ \frac{2}{x+1} + \frac{1}{8}\ \frac{\sqrt{x^2+2x-3}}{(x+1)^2} + C$

5. Evaluate : $\int \frac{(\cos\ 2x-3)}{\cos^4 x\sqrt{4-\cot^2 x}}\ dx$ को सरल कीजिए – **Ans.** $C - \frac{1}{3}\tan x.(2+\tan^2 x).\ \sqrt{4-\cot^2 x}$

Sol. $\int \frac{(\cos 2x-3)\quad dx}{\cos^4 x\sqrt{4-\cot^2 x}}$

$= -\,2.\ \int \frac{\sin^2 x+1}{\cos^2 x\sqrt{4\tan^2 x-1}}\ \tan x\ \sec^2 x\ dx$

Put $4\tan^2 x - 1 = t^2$, then $4\tan x \sec^2 x\, dx = t\, dt$

$$\Rightarrow \quad I = -2\int \frac{\frac{t^2+1}{2}+1}{t} \times \frac{t\ dt}{4} = -\frac{1}{4}\,t\left(\frac{t^2}{3}+3\right) + C = -\frac{1}{4}\sqrt{4\tan^2 x-1}\left(\frac{4\tan^2 x-1}{3}+3\right) + C$$

$$= -\frac{1}{3}\tan x\,\sqrt{4-\cot^2 x}\,(2+\tan^2 x) + C$$

Hindi. $\int \frac{(\cos 2x-3)\ dx}{\cos^4 x\sqrt{4-\cot^2 x}}$

$$= -2.\int \frac{\sin^2 x+1}{\cos^2 x\sqrt{4\tan^2 x-1}}\tan x \sec^2 x\, dx$$

माना $4\tan^2 x - 1 = t^2$ तो $4\tan x \sec^2 x\, dx = t\, dt$

$$\Rightarrow \quad I = -2\int \frac{\frac{t^2+1}{2}+1}{t} \times \frac{t\ dt}{4} = -\frac{1}{4}\,t\left(\frac{t^2}{3}+3\right) + C = -\frac{1}{4}\sqrt{4\tan^2 x-1}\left(\frac{4\tan^2 x-1}{3}+3\right) + C$$

$$= -\frac{1}{3}\tan x\,\sqrt{4-\cot^2 x}\,(2+\tan^2 x) + C$$

6. Evaluate : $\left[\frac{\sqrt{x^2+1}\,\{\ell n(x^2+1)-2\ell n x\}}{x^4}\right] dx$ को सरल कीजिए –

Ans. $\frac{2\ (x^2+1)\sqrt{x^2+1}}{9x^3}.\left[1-\frac{3}{2}\ell n\left(1+\frac{1}{x^2}\right)\right] + C$

Sol. $\int\left[\frac{\sqrt{x^2+1}\,\{\ell n\ (x^2+1)-2\ell n x\}}{x^4}\right] dx = \int\left[\sqrt{1+\frac{1}{x^2}}\left\{\ell n\ \left(1+\frac{1}{x^2}\right)\right\}\cdot\frac{1}{x^3}\right] dx$

Put $1+\frac{1}{x^2} = t^2$, then $-\frac{2}{x^3}\,dx = 2t\,dt$

$$= -\int 2t^2 \ell n\ t\ \ dt = -\frac{2t^3}{3}\ell n\ \ t + 2\int \frac{1}{t}\times\frac{t^3}{3}\,dt = -\frac{2t^3}{3}\ \ell n\, t + \frac{2t^3}{9} + C = \frac{2t^3}{3}\left(\frac{1}{3}-\ell n\ \ t\right) + C$$

$$= \frac{2(x^2+1)\sqrt{x^2+1}}{9x^3}\left[1-3\ \ \ell n\ \ \left(1+\frac{1}{x^2}\right)\right] + C$$

7. Evaluate : $\int \frac{x}{(7x-10-x^2)^{3/2}}\,dx$ को सरल कीजिए –

Ans. $\frac{2(7x-20)}{9\sqrt{7x-10-x^2}} + C$

Sol. taking $\sqrt{7x-10-x^2} = \sqrt{(5-x)(x-2)} = (5-x)\,t$ लेने पर

then तो $x = \frac{5t^2+2}{t^2+1}$ So अतः $dx = \frac{6t}{(t^2+1)^2}\,dt$

$$I = \int \frac{\frac{5t^2+2}{t^2+1}}{\left(\frac{3t}{t^2+1}\right)^3}\ \cdot\ \frac{6t}{(t^2+1)^2}\,dt$$

$$= \frac{2}{9}\int\left(\frac{5t^2+2}{t^2}\right)dt = \frac{10}{9}\,t - \frac{4}{9t} + C = \frac{2}{9}\left[5\ .\sqrt{\frac{x-2}{5-x}} - 2\sqrt{\frac{5-x}{x-2}}\right] + C = \frac{2(7x-20)}{9\sqrt{7x-10-x^2}} + C$$

8. If $\int \frac{x\cos\alpha + 1}{\left(x^2 + 2x\cos\alpha + 1\right)^{3/2}}\,dx = \frac{f(x)}{\sqrt{g(x)}} + C$ then find f(x) and g(x).

यदि $\int \frac{x\cos\alpha + 1}{\left(x^2 + 2x\cos\alpha + 1\right)^{3/2}}\,dx = \frac{f(x)}{\sqrt{g(x)}} + C$ हो तो f(x) तथा g(x) ज्ञात कीजिए।

Ans. $x;\ x^2 + 2x\cos\alpha + 1$

Sol. $I = \int \frac{(x\cos\alpha + 1)\ .\ dx}{(x^2 + 2x\cos\alpha + 1)^{3/2}} = \int \frac{x^3\left(x\frac{\cos\alpha}{x^3} + \frac{1}{x^3}\right)\ dx}{x^3\left(1 + \frac{2\cos\alpha}{x} + \frac{1}{x^2}\right)^{3/2}}$

Put $1 + \frac{2\cos\alpha}{x} + \frac{1}{x^2} = t$, then $\left(\frac{\cos x}{x^2} + \frac{1}{x^3}\right) dx = -\frac{dt}{2}$

$\Rightarrow \quad I = \int \frac{-dt/2}{t^{3/2}} = \frac{1}{t^{1/2}} + C = \frac{x}{\sqrt{x^2 + 2x\cos\alpha + 1}} + C$

comparing : $f(x) = x$ and, $g(x) = x^2 + 2x\cos\alpha + 1$

तुलना करने पर : $f(x) = x$ और, $g(x) = x^2 + 2x\cos\alpha + 1$

9. Evaluate : $\cos x.\ e^x.\ x^2\, dx$ को सरल कीजिए –

Ans. $\frac{1}{2} e^x [(x^2 - 1)\cos x + (x - 1)^2 . \sin x] + C$

Sol. $I = \int \cos x\ .\ e^x x^2\ dx$

$I = x^2 \int \cos x\ .\ e^x dx - \int 2x\ \left(\int \cos x . e^x dx\right)\ dx$

$\left(\because \int \cos x . e^x\ dx = \frac{e^x}{2}(\cos x + \sin x)\ \ \&\ \ \int \sin x\ .\ e^x dx = \frac{e^x}{2}(\sin x - \cos x)\right)$

$= x^2 \left[\frac{e^x}{2}(\cos x + \sin x)\right] - \int x\ .\ e^x(\cos x + \sin x)\ dx$

$= \frac{x^2 e^x}{2}(\cos x + \sin x) - \int x e^x \cos x\ dx - \int x e^x \sin x\ dx$

$= \frac{x^2 e^x}{2}(\cos x + \sin x) - \left(x\frac{e^x}{2}(\cos x + \sin x) - \int \frac{e^x}{2}(\cos x + \sin x)\ dx\right) - \left(x\frac{e^x}{2}(\sin x - \cos x) - \int \frac{e^x}{2}(\sin x - \cos x)\ dx\right)$

$= \frac{x^2 e^x}{2}(\cos x + \sin x) - x e^x \sin x + \int e^x \sin x\ dx$

$= \frac{x^2 e^x}{2}(\cos x + \sin x) - x e^x \sin x + \frac{e^x}{2}(\sin x - \cos x) + C$

$= \frac{e^x \cos x}{2}(x^2 - 1) + \frac{e^x \sin x}{2}(x^2 - 2x + 1) + C = \frac{e^x}{2}((x^2 - 1)\cos x + (x - 1)^2 \sin x) + C$

10. Evaluate : $\int e^x\left(\frac{x^3-x+2}{(x^2+1)^2}\right)dx$ को सरल कीजिए – **Ans.** $e^x\left(\frac{x+1}{x^2+1}\right)+C$

Sol. $\int e^x\left(\frac{x^3-x+2}{(x^2+1)^2}\right)dx = \int e^x\left(\frac{(x+1)(x^2+1)+(1-2x-x^2)}{(x^2+1)^2}\right)dx$

$= \int e^x\left(\frac{x+1}{(x^2+1)}+\frac{1-2x-x^2}{(x^2+1)^2}\right)dx = e^x\left(\frac{x+1}{(x^2+1)}\right)+C \quad \left(\because \frac{d}{dx}\left(\frac{x+1}{x^2+1}\right) = \frac{1-2x-x^2}{(x^2+1)^2}\right)$

11. Evaluate : $\int \frac{x^2}{(x\sin x+\cos x)^2}dx$ को सरल कीजिए –

Ans. $\frac{\sin x - x\cos x}{x\sin x+\cos x}+C$

Sol. $\int \frac{x^2\,dx}{(x\sin x+\cos x)^2} = \int \frac{x}{\cos x}\cdot\frac{x\cos x}{(x\sin x+\cos x)^2}dx$

$= \frac{x}{\cos x}\int\frac{x\cos x}{(x\sin x+\cos x)^2}dx - \int\left[\frac{d}{dx}(x\sec x)\int\frac{x\cos x}{(x\sin x+\cos x)^2}dx\right]dx$

$= \frac{x}{\cos x}\left(-\frac{1}{x\sin x+\cos x}\right) + \int \sec^2 x\,dx$

$\left(\because \int\frac{x\cos x}{(x\sin x+\cos x)^2}dx = \frac{-1}{x\sin x+\cos x}\right.$ and (और),

$\left.\frac{d}{dx}(x\sec x) = \sec x + x\sec x\tan x = \sec x\left(1+\frac{x\sin x}{\cos x}\right) = \sec^2 x(x\sin x+\cos x)\right)$

$= \frac{-x}{\cos x(x\sin x+\cos x)} + \frac{\sin x}{\cos x}+C$

$= \frac{-x+x\sin^2 x+\sin x\cos x}{\cos x(x\sin x+\cos x)} = \frac{\sin x - x\cos x}{x\sin x+\cos x}+C$

12. Evaluate : $\int \sin 4x\,.\,e^{\tan^2 x}\,dx$ को सरल कीजिए –

Ans. $-2\cos^4 x.\ e^{\tan^2 x}+C$

Sol. $\int \sin 4x\,.\,e^{\tan^2 x}\,dx = 4\int \sin x\cos x\cos 2x\ e^{\tan^2 x}dx =$

$4\int \tan x\,.\,\sec^2 x\,.\,\cos^4 x\,.\,\cos 2x\ e^{\tan^2 x}dx$

$= 2\int \frac{1}{(1+t)^2}\ \frac{1-t}{1+t}\ e^t\,dt$ (Putting, $\tan^2 x = t \Rightarrow 2\tan x\cdot\sec^2 x\,dx = dt$)

$= -2\int \frac{(t+1)-2}{(1+t)^3}e^t\,dt$

$= -2\int e^t\left(\frac{1}{(1+t)^2}+\frac{-2}{(1+t)^3}\right)dt = \frac{-2e^t}{(1+t)^2}+C \quad \left(\because \int e^t(f(t)+f'(t))\,dt = e^t f(t)\right)$

$= -2e^{\tan^2 x}\cos^4 x + C$

Hindi $\int \sin 4x\,.\,e^{\tan^2 x}\,dx = 4\int \sin x\cos x\cos 2x\ e^{\tan^2 x}dx$

$= 4\int \tan x\,.\,\sec^2 x\,.\,\cos^4 x\,.\,\cos 2x\ e^{\tan^2 x}dx$

$= 2\int \frac{1}{(1+t)^2}\ \frac{1-t}{1+t}\ e^t\,dt$ (माना $\tan^2 x = t \Rightarrow 2\tan x\cdot\sec^2 x\,dx = dt$)

$$= -2\int \frac{(t+1)-2}{(1+t)^3}e^t \quad dt$$

$$= -2\int e^t\left(\frac{1}{(1+t)^2}+\frac{-2}{(1+t)^3}\right) \quad dt \;=\; \frac{-2e^t}{(1+t)^2}+C \quad \left(\because \quad \int e^t(f(t)+f'(t)) \quad dt \;=\; e^t f(t)\right)$$

$$= -2e^{\tan^2 x}\cos^4 x + C$$

13. Evaluate : $\int \tan^{-1} x.\ \ell n\ (1+x^2)\ dx$.को सरल कीजिए –

Ans. $x \tan^{-1} x.\ \ell n\ (1+x^2) + (\tan^{-1} x)^2 - 2x \tan^{-1} x + \ell n\ (1+x^2) - \left(\ell n\sqrt{1+x^2}\right)^2 + C$

Sol. $\int \tan^{-1} x \ .\ \ell n(1+x^2) \quad dx$

$$= x \tan^{-1} x\ .\ \ell n(1+x^2) - \int \frac{x}{1+x^2}\ell n \ (1+x^2)\ dx \ - \int \frac{2x^2\tan^{-1}x}{(1+x^2)}dx$$

$$= x \tan^{-1}x\ \ell n\ (1+x^2) - \frac{1}{4}\ \ell n^2\ (1+x^2) - 2\int \tan^{-1}x \quad dx + 2\int \frac{\tan^{-1}x}{1+x^2} \quad dx$$

$$= x \tan^{-1}\ \ell n\ (1+x^2) - \frac{1}{4} \times 4\ \ell n^2\left(\sqrt{1+x^2}\right) - 2x \tan^{-1}x + \ell n\ (1+x^2) + (\tan^{-1}x)^2 + C$$

14. Evaluate : $\int e^x \dfrac{1+nx^{n-1}-x^{2n}}{\left(1-x^n\right)\sqrt{1-x^{2n}}}dx$ को सरल कीजिए –

Ans. $e^x\sqrt{\dfrac{1+x^n}{1-x^n}} + C$

Sol. $I = \int e^x\left(\dfrac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}\right)dx = \int e^x\left(\sqrt{\dfrac{1+x^n}{1-x^n}}+\dfrac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right)dx$

Let $f(x) = \sqrt{\dfrac{1+x^n}{1-x^n}} \Rightarrow f'(x) = \dfrac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}} \quad \therefore \quad I = e^x.\sqrt{\dfrac{1+x^n}{1-x^n}} + C$

Hindi. $\int e^x\left(\dfrac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}\right)dx = \int e^x\left(\sqrt{\dfrac{1+x^n}{1-x^n}}+\dfrac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right)dx =$ I (माना)

माना $f(x) = \sqrt{\dfrac{1+x^n}{1-x^n}} \Rightarrow f'(x) = \dfrac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}} \quad \therefore \quad I = e^x.\sqrt{\dfrac{1+x^n}{1-x^n}} + C$

15. Evaluate : $\int \cos 2x\ \ell n\ (1+\tan x)\ dx$ को सरल कीजिए –

Ans. $\dfrac{1}{2}\ [\sin 2x.\ell n(1+\tan x) - x + \ell n\ |\sin x + \cos x|] + C$

Sol. $\int \cos 2x\ \ell n\ (1+\tan x)\ dx$

$$= \frac{1}{2}\sin 2x\ \ell n\ (1+\tan x) - \frac{1}{2}\int \frac{\sec^2 x}{(1+\tan x)}\ \sin 2x\ dx = \frac{1}{2}\ \sin 2x\ \ell n\ (1+\tan x) - \int \frac{\tan x+1-1}{1+\tan x}dx$$

$$= \frac{1}{2}\ \sin 2x\ \ell n\ (1+\tan x) - x + \frac{1}{2}\int \frac{2\cos x+\sin x-\sin x}{\sin x+\cos x} \quad dx$$

$= \frac{1}{2}\sin 2x\,\ell n\,(1+\tan x) - x + \frac{x}{2} + \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$

$= \frac{1}{2}\,[\sin 2x\,\ell n\,(1+\tan x) - x + \ell n\,|\sin x + \cos x|] + C$

16. Evaluate : $\int \frac{dx}{(a+b\cos x)^2}$, (a > b) को सरल कीजिए –

Ans. $-\frac{b\sin x}{(a^2-b^2)(a+b\cos x)} + \frac{2a}{(a^2-b^2)^{3/2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2} + C$

Sol. $\int \frac{dx}{(a+b\cos x)^2}$, (a > b)

$I = -\frac{1}{b}\int \text{cosec}\ x\ .\ \frac{-b\sin x}{(a+b\ \cos x)^2}\,dx = \frac{1}{b}\,(\text{cosec}\ x)\,.\,\frac{1}{(a+b\cos x)} + \frac{1}{b}\int \frac{\text{cosec}\ x\ \ \cot x}{(a+b\cos x)}\,dx$

$= \frac{1}{b}\,(\text{cosec}\ x)\,.\,\frac{1}{(a+b\cos x)} + \frac{1}{b}\,I_1$

$I_1 = \int \frac{\cos\ x}{(1-\cos x)(1+\cos x)(a+b\cos x)}\,dx$

Use partial fractions and proceed
आंशिक भिन्नों का प्रयोग करके हल कीजिए।

17. Evaluate : $\int \frac{\sqrt{2-x-x^2}}{x^2}\,dx$ को सरल कीजिए –

Ans. $-\frac{\sqrt{2-x-x^2}}{x} + \frac{\sqrt{2}}{4}\ \ln\left|\frac{4-x+2\sqrt{2}\,\sqrt{2-x-x^2}}{x}\right| - \sin^{-1}\left(\frac{2x+1}{3}\right) + K$

Sol. $\int \frac{\sqrt{2-x-x^2}}{x^2}\,dx$

$= \sqrt{2-x-x^2}\ .\left(-\frac{1}{x}\right) + \int \frac{-(1+2x)}{2\sqrt{2-x-x^2}}\ .\ \frac{1}{x}\,dx$

$= -\frac{\sqrt{2-x-x^2}}{x} - \int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2}} - \frac{1}{2}\int \frac{1/x^2 dx}{\sqrt{\frac{2}{x^2} - \frac{1}{x} - 1}}$

$= -\frac{\sqrt{2-x-x^2}}{x} - \sin^{-1}\left(\frac{x+\frac{1}{2}}{3/2}\right) - I_1$

$I_1 = \int \frac{-dt}{2\sqrt{2t^2-t-1}}$ taking $\frac{1}{x} = t$, then $-\frac{1}{x^2}\,dx = dt$

$= -\frac{1}{2\sqrt{2}}\int \frac{dt}{\sqrt{\left(t-\frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2}} = -\frac{1}{2\sqrt{2}}\log\left|\left(t-\frac{1}{4}\right) + \sqrt{\left(t-\frac{1}{4}\right)^2 - \frac{9}{16}}\right|$

$= -\frac{1}{2\sqrt{2}}\log\left|\left(\frac{1}{x}-\frac{1}{4}\right) + \sqrt{\frac{1}{x^2} - \frac{1}{2x} - \frac{1}{2}}\right| = -\frac{1}{2\sqrt{2}}\log\left|\left(\frac{4-x}{4x}\right) + \frac{\sqrt{2-x-x^2}}{\sqrt{2}\ \ x}\right|$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{(4-x)+2\sqrt{2}\sqrt{2-x-x^2}}{4x}\right| + C = -\frac{1}{2\sqrt{2}}\log\left|\frac{(4-x)+2\sqrt{2}\sqrt{2-x-x^2}}{x}\right| + K$$

Hindi $\int \frac{\sqrt{2-x-x^2}}{x^2}\,dx$

$$= \sqrt{2-x-x^2}\cdot\left(-\frac{1}{x}\right) + \int \frac{-(1+2x)}{2\sqrt{2-x-x^2}}\cdot\frac{1}{x}\,dx$$

$$= -\frac{\sqrt{2-x-x^2}}{x} - \int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2}} - \frac{1}{2}\int \frac{1/x^2 dx}{\sqrt{\frac{2}{x^2} - \frac{1}{x} - 1}}$$

$$= -\frac{\sqrt{2-x-x^2}}{x} - \sin^{-1}\left(\frac{x+\frac{1}{2}}{3/2}\right) - I_1$$

$I_1 = \int \frac{-dt}{2\sqrt{2t^2 - t - 1}}$ माना $\frac{1}{x} = t$, तो $-\frac{1}{x^2}dx = dt$

$$= -\frac{1}{2\sqrt{2}}\int \frac{dt}{\sqrt{\left(t-\frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2}} = -\frac{1}{2\sqrt{2}}\log\left|\left(t-\frac{1}{4}\right) + \sqrt{\left(t-\frac{1}{4}\right)^2 - \frac{9}{16}}\right|$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\left(\frac{1}{x}-\frac{1}{4}\right) + \sqrt{\frac{1}{x^2} - \frac{1}{2x} - \frac{1}{2}}\right| = -\frac{1}{2\sqrt{2}}\log\left|\left(\frac{4-x}{4x}\right) + \frac{\sqrt{2-x-x^2}}{\sqrt{2}\;x}\right|$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{(4-x)+2\sqrt{2}\sqrt{2-x-x^2}}{4x}\right| + C = -\frac{1}{2\sqrt{2}}\log\left|\frac{(4-x)+2\sqrt{2}\sqrt{2-x-x^2}}{x}\right| + K$$

18. Integrate: $\int \frac{(5x^2 - 12)\,dx}{(x^2 - 6x + 13)^2}$ को सरल कीजिए –

Ans. $\frac{13x - 159}{8(x^2 - 6x + 13)} + \frac{53}{16}\tan^{-1}\frac{x-3}{2} + C$

Sol: $5x^2 - 12 = l(x^2 - 6x + 13) + m(2x - 6) + n$

equating co-efficient गुणांकों की तुलना करने पर $l = 5$; $m = 15$; $n = 13$

$$\Rightarrow \quad I = \int \frac{5\,dx}{(x-3)^2 + 4} + \int \frac{15(2x-6)}{(x^2 - 6x + 13)^2}dx + 13\int \frac{dx}{\left[(x-3)^2 + 4\right]^2}$$

Now अब $I_1 = \frac{5}{2}\tan^{-1}\frac{x-3}{2}$; $I_2 = -\frac{15}{x^2 - 6x + 13}$ & $I_3 = 13\int \frac{2\sec^2\theta}{16\sec^4\theta}\,d\theta$,

where जहाँ $x - 3 = 2\tan\theta$, $= \frac{13}{16}\int(1 + \cos 2\theta)\,d\theta = \frac{13}{16}\left[\theta + \frac{1}{2}\sin 2\theta\right]$

$$= \frac{13}{16}\left[\tan^{-1}\frac{x-3}{2}\right] + \frac{13}{32}\cdot\frac{\frac{x-3}{1}}{1+\left(\frac{x-3}{2}\right)^2} = \frac{13}{16}\tan^{-1}\frac{x-3}{2} + \frac{13}{8}\cdot\frac{x-3}{x^2 - 6x + 9}$$

$$\Rightarrow \quad I = I_1 + I_2 + I_3 = \frac{13x - 159}{8(x^2 - 6x + 13)} + \frac{53}{16}\tan^{-1}\frac{x-3}{2} + C$$

19. If $\int \frac{3x^2+2x}{x^6+2x^5+x^4+2x^3+2x^2+5}$ dx = F(x), then find the value of [F(1) – F(0)], where [.] represents greatest integer function.

यदि $\int \frac{3x^2+2x}{x^6+2x^5+x^4+2x^3+2x^2+5}$ dx = F(x) है, तो [F(1) – F(0)] का मान ज्ञात कीजिए ,जहाँ [.] महत्तम पूर्णांक फलन को प्रदर्शित करता है। .

Ans. 0

Sol. $\int \frac{3x^2+2x}{x^6+2x^5+x^4+2x^3+2x^2+5}\, dx = \int \frac{3x^2+2x}{(x^3+x^2+1)^2+4}\, dx = \frac{1}{2}\tan^{-1}\left(\frac{x^3+x^2+1}{2}\right) + C$

$\therefore \quad F(x) = \frac{1}{2}\tan^{-1}\left(\frac{x^3+x^2+1}{2}\right) + C$

$\therefore \quad F(1) - F(0) = \frac{1}{2}\left(\tan^{-1}\frac{3}{2} - \tan^{-1}\frac{1}{2}\right) = \frac{1}{2}\tan^{-1}\frac{4}{7}$

$\therefore \quad 0 < F(1) - F(0) < \frac{1}{2} \cdot \frac{\pi}{2} < 1$

$\therefore \quad [F(1) - F(0)] = 0$

20. Evaluate : $\int \frac{\ell n\ (1 + \sin^2 x)}{\cos^2 x}$ dx. को सरल कीजिए –

Ans. $\tan x \ln(1 + \sin^2 x) - 2x + \sqrt{2}\tan^{-1}(\sqrt{2}.\tan x) + C.$

Sol. $\int \frac{\ell n(1+\sin^2 x)\ \ dx}{\cos^2 x}$

$I = \ell n(1 + \sin^2 x) . \tan x - 2\int \frac{\sin^2 x + 1 - 1}{1+\sin^2 x}\, dx = \ell n(1 + \sin^2 x) . \tan x - 2x + 2\int \frac{\sec^2 x\ \ dx}{1+2\tan^2 x}$

$= \ell n(1 + \sin^2 x) . \tan x - 2x + \sqrt{2}\tan^{-1}(\sqrt{2}\tan x) + C$

21. Evaluate : $\int \frac{1+\cos\alpha\ \cos x}{\cos\alpha + \cos x}$ dx को सरल कीजिए –

Ans. $x\cos\alpha + \sin\alpha\ \ell n\left|\frac{\cos\frac{1}{2}(\alpha - x)}{\cos\frac{1}{2}(\alpha + x)}\right| + C$

Sol. $I = \int \frac{1+\cos\alpha\cos x}{\cos\alpha+\cos x}\, dx = \int \frac{1+\cos\alpha\cos x+\cos^2\alpha-\cos^2\alpha}{\cos\alpha+\cos x}\, dx$

$= \int \frac{\cos\alpha(\cos x+\cos\alpha)}{\cos\alpha+\cos x}\, dx + \int \frac{\sin^2\alpha}{\cos\alpha+\cos x}\, dx = x\cos\alpha + \sin^2\alpha . I_1$

$I_1 = \int \frac{\sec^2 x/2}{\left(1+\tan^2\frac{x}{2}\right)\cos\alpha + \left(1-\tan^2\frac{x}{2}\right)}\, dx = \int \frac{\sec^2\frac{x}{2}}{(1+\cos\alpha)-(1-\cos\alpha)\tan^2\frac{x}{2}}\, dx$

$= \frac{1}{\sin\alpha}\ \ell n\left|\left(\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\tan\frac{x}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\tan\frac{x}{2}}\right)\right| + C = \frac{1}{\sin\alpha}\ \ell n\left|\left(\frac{\cos\frac{\alpha}{2}\cos\frac{x}{2}+\sin\frac{\alpha}{2}\sin\frac{x}{2}}{\cos\frac{\alpha}{2}\cos\frac{x}{2}-\sin\frac{\alpha}{2}\sin\frac{x}{2}}\right)\right| + C$

Hindi $I = \int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(x-\beta)}}$ $\quad (x-\alpha) = \frac{1}{t}$ रखने पर $\Rightarrow$ $dx = -\frac{1}{t^2}\,dt$

$\Rightarrow$ $I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\cdot\frac{1}{\sqrt{t}}\sqrt{(\alpha-\beta)+\frac{1}{t}}} = -\int \frac{dt}{\sqrt{1+(\alpha-\beta)\ t}}$

माना $1 + (\alpha-\beta)\,t = z^2$, तो $(\alpha-\beta)\,dt = 2z\,dz$

$\Rightarrow$ $I = -\frac{2}{(\alpha-\beta)}\int \frac{zdz}{z} = -\left(\frac{2}{\alpha-\beta}\right)\sqrt{1+\frac{\alpha-\beta}{x-\alpha}} + C = -\left(\frac{2}{\alpha-\beta}\right)\sqrt{\frac{x-\beta}{x-\alpha}} + C$

वैकल्पिक : $x = \alpha\sec^2\theta - \beta\tan^2\theta$ रखने पर

वैकल्पिक : $\frac{x-\beta}{x-\alpha} = t^2$ रखने पर

वैकल्पिक : $\frac{x-\beta}{x-\alpha} = z$ रखने पर

24. Evaluate $\int \frac{(\cos 2x)^{1/2}}{\sin x}dx$ को सरल कीजिए –

Ans. $\sqrt{2}\log\left[\frac{\sqrt{\cot^2 x-1}+\sqrt{2\cot^2 x}}{\sqrt{\cot^2 x+1}}\right] - \log\left[\cot x+\sqrt{\cot^2 x-1}\right]+c$

Sol. Let $I = \int \frac{(\cos 2x)^{1/2}}{\sin x}dx = \int \sqrt{\frac{\cos^2 x-\sin^2 x}{\sin^2 x}}\,dx = \sqrt{\cot^2 x-1}\,dx$ put $\sqrt{\cot^2 x-1} = y$

$\cot^2 x = y^2 + 1$

$\Rightarrow$ $-2\cot x.\text{cosec}^2 x.dx = 2y.dy$

$dx = \frac{-ydy}{\cot x(1+\cot^2 x)} = \frac{-ydy}{\sqrt{1+y^2}\,(2+y^2)}$

so that $I = \int \frac{y\times y.dy}{\sqrt{1+y^2}\,(2+y^2)} = -\int \frac{(y^2+2+2).dy}{(y^2+2)\sqrt{y^2+1}}$

$= -\int \frac{1}{\sqrt{y^2+1}}.dy + 2 - \int \frac{dy}{(y^2+2)\sqrt{y^2+1}} = -\log[y + \sqrt{y^2+1}] + 2I_1$

$I_1 = \int \frac{dy}{(y^2+2)\sqrt{y^2+1}}$

put $\frac{y^2+1}{y^2+2} = t^2$

$2y.dy = \frac{(1-t^2)(4t)-(2t^2-1)(-2t)}{(1-t^2)^2}$

$y.dy = \frac{t}{(1-t^2)^2}$

$\Rightarrow$ $dy = \frac{t\sqrt{1-t^2}}{(1-t^2)^2\sqrt{2t^2-1}}$

$y^2 + 2 = \frac{2t^2-1}{1-t^2}+2 = \frac{2t^2-1+2-2t^2}{1-t^2} = \frac{1}{1-t^2}$ and $y^2 + 1 = \frac{2t^2-1}{1-t^2}+1 = \frac{2t^2-1+1-t^2}{1-t^2} = \frac{t^2}{1-t^2}$

$\therefore \quad I_1 = \int \frac{t\sqrt{1-t^2}}{(1-t^2)^2\sqrt{2t^2-1}} \frac{\sqrt{1-t^2}}{t}.(1-t^2)dt = \int \frac{dt}{\sqrt{2t^2-1}} = \frac{1}{\sqrt{2}}\log(\sqrt{2}t + \sqrt{2t^2-1})$

$= \frac{1}{\sqrt{2}}\log\left(\sqrt{\frac{2(y^2+1)}{y^2+2}}\right) + \frac{y}{\sqrt{y^2+2}} = \frac{1}{\sqrt{2}}\log\frac{y+2\sqrt{(y^2+1)}}{\sqrt{y^2+2}}$

$\therefore \quad I = -\log[y + \sqrt{y^2+1}\ ] + \sqrt{2}\ \log\left[\frac{y+\sqrt{2(y^2+1)}}{\sqrt{y^2+2}}\right] + c$

$= \sqrt{2}\log\left[\frac{\sqrt{\cot^2 x - 1} + \sqrt{2\cot^2 x}}{\sqrt{\cot^2 x + 1}}\right] - \log\left[\cot x + \sqrt{\cot^2 x - 1}\right] + c$

Aliter : $I = \int \frac{1-2\sin^2 x}{\sin x.\sqrt{\cos 2x}}dx$

$= \int \frac{dx}{\sin x\sqrt{\cos^2 x - \sin^2 x}} - 2\int \frac{\sin x}{\sqrt{2\cos^2 x - 1}}dx = \int \frac{-dy}{\sqrt{u^2-1}} + 2\int \frac{dt}{\sqrt{2t^2-1}}$

In the 1st integral, put cot x = t $\Rightarrow$ –cosec²dx = dt
In the 2nd integral put cosx = t
–sinxdx = dt

Aliter : $I = \int \frac{1-2\sin^2 x}{\sin x.\sqrt{\cos 2x}}dx$

$= \int \frac{dx}{\sin x\sqrt{\cos^2 x - \sin^2 x}} - 2\int \frac{\sin x}{\sqrt{2\cos^2 x - 1}}dx = \int \frac{-dy}{\sqrt{u^2-1}} + 2\int \frac{dt}{\sqrt{2t^2-1}}$

प्रथम समाकलन में cot x = t रखने पर $\Rightarrow$ –cosec²dx = dt
दूसरे समाकलन में cosx = t रखने पर
–sinxdx = dt

25. Evaluate $\int \frac{\sin^3 \frac{x}{2}}{\cos\frac{x}{2}\sqrt{\cos^3 x + \cos^2 x + \cos x}}dx$ को सरल कीजिए –

Ans. $\sec^{-1}\left(\sqrt{\cos x} + \frac{1}{\sqrt{\cos x}}\right) + c.$

Sol. $\int \frac{\sin^3 \frac{x}{2}}{\cos\frac{x}{2}\sqrt{\cos^3 x + \cos^2 x + \cos x}}dx = \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}\sqrt{\cos^3 x + \cos^2 x + \cos x}}dx$

$= \int \frac{\sin x(1-\cos x)}{2(1+\cos x)\sqrt{\cos^3 x + \cos^2 x + \cos x}}$

$= \int \frac{(1-t)(-dt)}{2(1+t)\sqrt{t^3+t^2+t}}$ {where t = cos x}

$= \frac{1}{2}\int \frac{t-1}{(t+1)\ \sqrt{t^3+t^2+t}}dt$

$= \int \frac{(u^2-1)\ du}{(u^2+1)\sqrt{u^4+u^2+1}}$ {where $\sqrt{t} = u$ }

$$= \int \frac{\left(1-\frac{1}{u^2}\right)\,du}{\left(u+\frac{1}{u}\right)\sqrt{u^2+\frac{1}{u^2}+1}}$$

$$= \int \frac{dv}{v\sqrt{v^2-1}} = \sec^{-1}|v| + c. \qquad \left\{\text{where } u+\frac{1}{u} = v\right\}$$

$$= \sec^{-1}\left(u+\frac{1}{u}\right) + c \qquad = \sec^{-1}\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right) + c \quad = \sec^{-1}\left(\sqrt{\cos x}+\frac{1}{\sqrt{\cos x}}\right) + c.$$

26. If $\int \frac{x^2}{x^4+3x^2+9}dx = A\tan^{-1}\left(\frac{x^2-3}{3x}\right) + \frac{B}{\sqrt{3}}\ell n\left|\frac{x^2-\sqrt{3}\ x+3}{x^2+\sqrt{3}\ x+3}\right| + c$, then find the value of 12(A + B).

यदि $\int \frac{x^2}{x^4+3x^2+9}dx = A\tan^{-1}\left(\frac{x^2-3}{3x}\right) + \frac{B}{\sqrt{3}}\ell n\left|\frac{x^2-\sqrt{3}\ x+3}{x^2+\sqrt{3}\ x+3}\right| + c$ है, तो 12(A + B) का मान ज्ञात कीजिए।

Ans. 5

Sol. $\int \frac{x^2}{x^4+3x^2+9}dx = \frac{1}{2}\int \frac{x^2+3+x^2-3}{x^4+9+3x^2}dx$

$$= \frac{1}{2}\int \frac{d\left(x-\frac{3}{x}\right)}{\left(x-\frac{3}{x}\right)^2+9} + \frac{1}{2}\int \frac{d\left(x+\frac{3}{x}\right)}{\left(x+\frac{3}{x}\right)^2-3} = \frac{1}{6}\tan^{-1}\frac{x^2-3}{3x} + \frac{1}{4\sqrt{3}}\ell n\left|\frac{x^2-\sqrt{3}\ x+3}{x^2+\sqrt{3}\ x+3}\right| + c$$

$\therefore \quad A = \frac{1}{6},\ B = \frac{1}{4} \qquad \therefore \quad 12(A + B) = 5$

27. Evaluate $\int \frac{3\cos x+2}{\sin x+2\cos x+3}dx$ को सरल कीजिए –

Ans. $\frac{6}{5}x+\frac{3}{5}\log|\sin x+2\cos x+3|-\frac{8}{5}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)+C$

Sol. $3\cos x + 2 = A(\sin x + 2\cos x + 3) + B\frac{d}{dx}(\sin x + 2\cos x + 3) + \lambda$

$3\cos x + 2 = \sin x\,(A - 2B) + \cos x(2A + B) + 3A + \lambda$

Comparing the coefficients of sinx, cosx and constant terms of both sides, we get

sinx, cosx और अचर पदों तुलना करने पर

$A - 2B = 0,\ 2A + B = 3,\ 3A + \lambda = 2$

Solving these equation we get समीकरण को हल करने पर $A = \frac{6}{5}$, $B = \frac{3}{5}$, $\lambda = -\frac{8}{5}$

$$= \int \frac{3\cos x+2}{\sin x+2\cos x+3}dx = \int \frac{\frac{6}{5}(\sin x+2\cos x+3)+\frac{3}{5}(\cos x-2\sin x)-\frac{8}{5}}{\sin x+2\cos x+3}$$

$$= \int \frac{3\cos x+2}{\sin x+2\cos x+3}dx = \int \frac{A(\sin x+2\cos x+3)+B(\cos x-2\sin x)+\lambda}{\sin x+2\cos x+3}$$

$$= \frac{6}{5}\times\frac{3}{5}\log|\sin x+2\cos x| - \frac{8}{5}\int\frac{1}{\sin x+2\cos x+3}$$

Now $\int \frac{1}{\sin x + 2\cos x + 3} dx$

Put sinx = $\frac{2\tan x/2}{1+\tan^2 x/2}, \cos x = \frac{1-\tan^2 x/2}{1+\tan x/2}$ and $\tan\frac{x}{2} = t$

$\Rightarrow \quad \tan\frac{x}{2} = t \quad \Rightarrow \quad \frac{x}{2}\sec^2\frac{x}{2}\, dx = dt$

$$\int \frac{1}{\sin x + 2\cos x + 3} dx = \int \frac{2dt}{t^2+2t+5} = 2\int \frac{dt}{(t+1)^2+2^2} = \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right) + C$$

$$\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3} dx = \frac{6}{5}x + \frac{3}{5}\log|\sin x + 2\cos x + 3| - \frac{8}{5}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right) + C$$

28. Evaluate $\int \sqrt[3]{\tan x}\, dx$ को सरल कीजिए –

Ans. $-\frac{1}{2}\log(1+\tan^{2/3}x) + \frac{1}{4}\log(\tan^{4/3}x - \tan^{2/3}x + 1) + \frac{\sqrt{3}}{2}\tan^{-1}\frac{2\tan^{2/3}x - 1}{\sqrt{3}} + c$

Sol. Let माना $\tan x = z^{3/2}$, then तब $\sec^2 x dx = \frac{3}{2} z^{1/2} dz$

$\therefore \quad I = \int \sqrt[3]{z^{3/2}}\, \frac{3}{2} z^{1/2} \frac{1}{\sec^2 x} dz = \frac{3}{2}\int \frac{z}{1+z^3} dz = \frac{3}{2}\int \frac{z dz}{(1+z)(1-z+z^2)}$

Let माना $\frac{z}{(1+z)(1-z+z^2)} = \frac{A}{1+z} + \frac{Bz+C}{z^2-z+1} \Rightarrow \quad z \equiv A(z^2 - z + 1) + (Bz + C)(1 + z)$

put $z = -1$ रखने पर तब .Then $-1 = A(1 + 1 + 1)$ soअतः, $A = -1/3$

Equating coefficients of z^2 on both sides, $O = A + C$, $\therefore$ $C = 1/3$ and $B = 1/3$

z^2 के गुणाकों की तुलना करने पर $O = A + C$, $\therefore$ $C = 1/3$ तथा $B = 1/3$

$\therefore \quad I = \frac{3}{2}\int -\frac{1}{3}.\frac{dz}{1+z} + \frac{3}{2}\int -\frac{\frac{1}{3}z + \frac{1}{3}}{z^2-z+1} dz = -\frac{1}{2}\log(1+z) + \frac{1}{2}\int \frac{z+1}{z^2-z+1} dz$

$= -\frac{1}{2}\log(1+z) + \frac{1}{4}\int \frac{(2z-1)+3}{z^2-z+1} dz$

$= -\frac{1}{2}\log(1+z) + \frac{1}{4}\int \frac{(2z-1)dz}{z^2-z+1} + \int \frac{dz}{\left(z-\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= -\frac{1}{2}\log(1+z) + \frac{1}{4}\log(z^2 - z + 1) + \frac{3}{4}.\frac{2}{\sqrt{3}}\tan^{-1}\frac{z-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$

$= -\frac{1}{2}\log(1+\tan^{2/3}x) + \frac{1}{4}\log(\tan^{4/3}x - \tan^{2/3}x + 1) + \frac{\sqrt{3}}{2}\tan^{-1}\frac{2\tan^{2/3}x - 1}{\sqrt{3}} + c$

29. Evaluate : $\int \sqrt{\dfrac{\text{cosec } x - \cot x}{\text{cosec } x + \cot x}} \cdot \dfrac{\sec x}{\sqrt{1+2\sec x}}$ dxको सरल कीजिए –

Ans. $\sin^{-1}\left(\dfrac{1}{2}\sec^2\dfrac{x}{2}\right) + C$

Sol. $I = \int \sqrt{\dfrac{\text{cosec} x - \cot x}{\text{cosec} x + \cot x}} \cdot \dfrac{\sec x \ dx}{\sqrt{1+2\sec x}}$

$= \int \sqrt{\dfrac{1-\cos x}{1+\cos x}} \cdot \dfrac{\sec x \ dx}{\sqrt{1+2\sec x}} = \int \tan\dfrac{x}{2} \cdot \dfrac{dx}{\sqrt{\cos x}\sqrt{\cos x+2}}$

$= \int \tan\dfrac{x}{2} \cdot \dfrac{\sec^2\frac{x}{2} \ dx}{\sqrt{1-\tan^2\frac{x}{2}}\sqrt{3+\tan^2\frac{x}{2}}} = \int \dfrac{\tan\frac{x}{2}\sec^2\frac{x}{2}dx}{\sqrt{2-\sec^2\frac{x}{2}}\sqrt{2+\sec^2\frac{x}{2}}}$

$= \int \dfrac{\tan\frac{x}{2}\sec^2\frac{x}{2}dx}{\sqrt{2^2-\left(\sec^2\frac{x}{2}\right)^2}}$ Let $\sec^2\dfrac{x}{2} = t$, then $\sec^2\dfrac{x}{2}\tan\dfrac{x}{2}\,dx = dt$

$\Rightarrow \quad I = \int \dfrac{dt}{\sqrt{2^2-t^2}} = \sin^{-1}\left(\dfrac{t}{2}\right) + C = \sin^{-1}\left(\dfrac{1}{2}\sec^2\dfrac{x}{2}\right) + C$

Hindi $I = \int \sqrt{\dfrac{\text{cosec} x - \cot x}{\text{cosec} x + \cot x}} \cdot \dfrac{\sec x \ dx}{\sqrt{1+2\sec x}}$

$= \int \sqrt{\dfrac{1-\cos x}{1+\cos x}} \cdot \dfrac{\sec x \ dx}{\sqrt{1+2\sec x}} = \int \tan\dfrac{x}{2} \cdot \dfrac{dx}{\sqrt{\cos x}\sqrt{\cos x+2}}$

$= \int \tan\dfrac{x}{2} \cdot \dfrac{\sec^2\frac{x}{2} \ dx}{\sqrt{1-\tan^2\frac{x}{2}}\sqrt{3+\tan^2\frac{x}{2}}} = \int \dfrac{\tan\frac{x}{2}\sec^2\frac{x}{2}dx}{\sqrt{2-\sec^2\frac{x}{2}}\sqrt{2+\sec^2\frac{x}{2}}}$

$= \int \dfrac{\tan\frac{x}{2}\sec^2\frac{x}{2}dx}{\sqrt{2^2-\left(\sec^2\frac{x}{2}\right)^2}}$ माना $\sec^2\dfrac{x}{2} = t$, तो $\sec^2\dfrac{x}{2}\tan\dfrac{x}{2}\,dx = dt$

$\Rightarrow \quad I = \int \dfrac{dt}{\sqrt{2^2-t^2}} = \sin^{-1}\left(\dfrac{t}{2}\right) + C = \sin^{-1}\left(\dfrac{1}{2}\sec^2\dfrac{x}{2}\right) + C$